数学A

(1) $\cos n\theta + i \sin n\theta = e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$

(2) $\dfrac{1}{\sqrt{1+x}} = 1 - \dfrac{1}{2}x + \dfrac{3}{8}x^2 - \dfrac{15}{48}x^3$

(3) (i)

$$\int_0^\pi e^x \sin x\,dx = \left[\cancel{e^x \sin x}\right]_0^\pi - \int_0^\pi e^x \cos x\,dx$$

$$= -\left[e^x \cos x\right]_0^\pi - \int_0^\pi e^x \sin x\,dx$$

$$= -(-e^\pi - 1) - \int_0^\pi e^x \sin x\,dx$$

$$\therefore \int_0^\pi e^x \sin x\,dx = \frac{e^\pi + 1}{2} \quad //$$

(ii) $x = \cos\theta$ とおく

$dx = -\sin\theta\,d\theta$

x	$0 \to 1$
θ	$\frac{\pi}{2} \to 0$

$$\int_0^1 x\sqrt{1-x^2}\,dx = \int_{\frac{\pi}{2}}^0 -\cos\theta \sin^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta\,d\theta$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \int_0^{\frac{\pi}{2}} \underline{\frac{1-\cos 2\theta}{2}\cos\theta}\,d\theta \quad (*)$$

(*) の部分

$\frac{1}{2}\cos\theta - \frac{1}{2}\cos 2\theta\cos\theta$

$= \frac{1}{2}\cos\theta - \frac{1}{4}(\cos 3\theta + \cos\theta)$

$= \frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta \quad d\theta$$

$$= \left[\frac{1}{4}\sin\theta\right]_0^{\frac{\pi}{2}} - \left[\frac{1}{12}\sin 3\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3} \;_{//}$$

(4)

$$A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$$

(i) 固有値方程式 $\det(A - \lambda E) = 0$ より.

$$(-1-\lambda)^2 - 9 = 0$$
$$\lambda^2 + 2\lambda + 1 - 9 = 0$$
$$\lambda^2 + 2\lambda - 8 = 0$$
$$(\lambda + 4)(\lambda - 2) = 0$$

$$\lambda = 2, -4 \;_{//}$$

固有値は $2, -4$ //

$\lambda = 2$ のとき.

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left.\begin{aligned} -3x + 3y &= 0 \\ 3x - 3y &= 0 \end{aligned}\right\} \to x = y.$$

固有値ベクトルは. $t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
t: 任意

$\lambda = -4$ のとき

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left.\begin{aligned} &3x + 3y \\ &3x + 3y \end{aligned}\right\} \to x = -y$$

固有値ベクトルは
t: 任意

$$t\begin{pmatrix} 1 \\ -1 \end{pmatrix} \;_{//}$$

(ii) $P' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ とすると. $P'^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$D = P'^{-1} A P'$

$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} 2 & 2 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$ //

また、
直交行列 $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ //

(iii) $Q = -x_1^2 + 6x_1x_2 + x_2^2$

$A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$ $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

とすると.

$Q = \mathbf{x}^T A \mathbf{x}$

$= \mathbf{x}^T P D P^{-1} \mathbf{x}$

$= (P^T \mathbf{x})^T D (\underline{P^{-1}} \mathbf{x})$

$= (P^T \mathbf{x})^T D (\underline{P^T} \mathbf{x})$ 直交行列なので

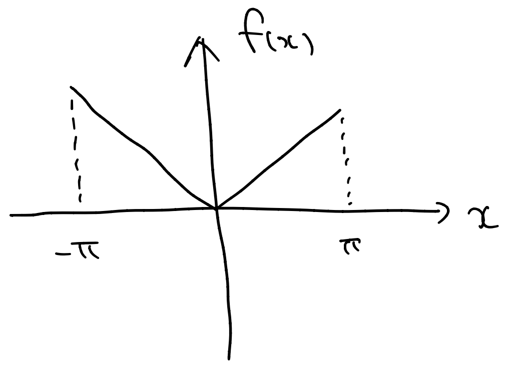
$= \mathbf{y}^T D \mathbf{y}$ $\mathbf{y} = P^T \mathbf{x}$ とすると.

$= (y_1, y_2)\begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$= 2y_1^2 - 4y_2^2$

ただし、$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 \\ \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2 \end{pmatrix}$

(5) (i)



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cos nx \, dx$$

$n=0$ のとき

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx = \frac{\cancel{2}}{\pi}\left[\frac{1}{\cancel{2}}x^2\right]_0^{\pi} = \pi$$ //

$n \neq 0$ のとき

$$\int_0^{\pi} x \cos nx \, dx = \left[\frac{x}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nx \, dx$$

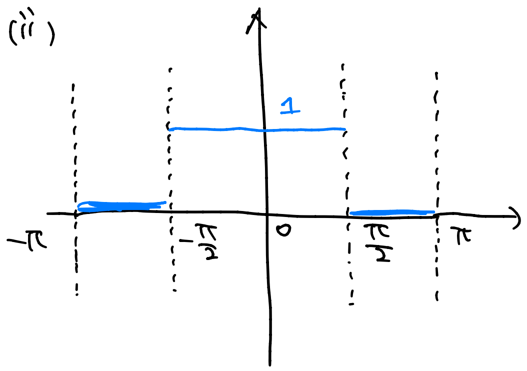
$$= 0 + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi}$$

$$= \frac{1}{n^2}\left\{(-1)^n - 1\right\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx \, dx = 0.$$

$$\therefore \quad f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2}\left\{(-1)^n - 1\right\}\cos nx$$ //

(ii)



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} \cos nx \; dx$$

$$= \begin{cases} n=0 \text{ のとき} \\ \dfrac{2}{\pi}\displaystyle\int_{0}^{\frac{\pi}{2}} dx = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1 \\ \\ n\neq 0 \text{ のとき} \\ \dfrac{2}{\pi}\left[\dfrac{1}{n}\sin nx\right]_{0}^{\pi/2} = \dfrac{2}{n\pi}\sin\dfrac{n\pi}{2} \end{cases}$$

$$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} \overset{\text{偶}}{f(x)}\ \underset{\text{奇}}{\sin nx}\ dx = 0$$

$$\therefore\ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx$$

//

数学B

(1) $$l = \int_0^T \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}\, ds$$

$$\begin{cases} \dfrac{dx}{ds} = -a\sin s \\ \dfrac{dy}{ds} = a\cos s \\ \dfrac{dz}{ds} = b \end{cases}$$

以上より

$$l = \int_0^T \sqrt{a^2\sin^2 s + a^2\cos^2 s + b^2}\, ds$$

$$= \int_0^T \sqrt{a^2+b^2}\, ds$$

$$= \sqrt{a^2+b^2}\, T \;//$$

(2) (i) $$\frac{\partial u}{\partial x} = \frac{-2yz}{(x^2+y^2)^2} + \frac{8x^2yz}{(x^2+y^2)^3}$$

$$= \frac{-2yz(x^2+y^2) + 8x^2yz}{(x^2+y^2)^3}$$

$$= \frac{6x^2yz - 2y^3z}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial y} = \frac{-2xz}{(x^2+y^2)^2} + \frac{8xy^2z}{(x^2+y^2)^3}$$

$$= \frac{-2xz(x^2+y^2) + 8xy^2z}{(x^2+y^2)^3}$$

$$= \frac{6xy^2z - 2xy^2z}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial z} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xz}{(x^2+y^2)^2} - \frac{4x(x^2-y^2)z}{(x^2+y^2)^3}$$

$$= \frac{2xz(x^2+y^2)}{(x^2+y^2)^3} - \frac{4xz(x^2-y^2)}{(x^2+y^2)^3} = \frac{-2x^3z+6xzy^2}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{-2yz}{(x^2+y^2)^2} - \frac{4y(x^2-y^2)z}{(x^2+y^2)^3}$$

$$= \frac{-2yz(x^2+y^2)}{(x^2+y^2)^3} - \frac{4yz(x^2-y^2)}{(x^2+y^2)^3} = \frac{-6x^2yz+2y^3z}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial z} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial w}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial w}{\partial y} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial w}{\partial z} = 0$$

(ii)

$$\mathrm{div}\, v = \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}\right)$$ ∴ 詳しい成分は(i)を参照

$$\mathrm{rot}\, v = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ u & v & w \\ e_x & e_y & e_z \end{vmatrix}$$

$$= (\partial_y w - \partial_z v)\, e_x + (\partial_z u - \partial_x w)\, e_y + (\partial_x v - \partial_y u)\, e_z$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}\, e_x + \left[\frac{-2xy}{(x^2+y^2)^2} - \frac{-2xy}{(x^2+y^2)^2}\right] e_y$$

$$+ \left[\frac{-2x^3z+6xzy^2}{(x^2+y^2)^3} - \frac{6xy^2z-2xy^2z}{(x^2+y^2)^3}\right] e_z$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}\, e_x$$ e_x : x の単位ベクトル.

(3) (i) $d\tau = \omega\, dt$ なので

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \omega \frac{dx}{d\tau}$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \omega \frac{d}{dt}\left(\frac{dx}{d\tau}\right) = \omega \frac{d\tau}{dt}\frac{d}{d\tau}\left(\frac{dx}{d\tau}\right) = \omega^2 \frac{d^2x}{d\tau^2}$$

以上より

$$\omega^2 \frac{d^2x}{d\tau^2} + x + \varepsilon x^3 = 0 \quad —— \ (*)$$

(ii)

$$\left.\begin{aligned} x &= x_0 + \varepsilon x_1 + \cdots \\ \omega &= 1 + \varepsilon \omega_1 + \cdots \end{aligned}\right\} \text{を} (*) \text{に代入する.}$$

$$(1+\varepsilon\omega_1+\cdots)^2\left(\frac{d^2x_0}{d\tau^2} + \varepsilon\frac{d^2x_1}{d\tau^2} + \cdots\right)$$
$$+(x_0+\varepsilon x_1+\cdots) + \varepsilon(x_0+\varepsilon x_1+\cdots)^3 = 0.$$

ε の0次に注目すると,

$$\frac{d^2x_0}{d\tau^2} + x_0 = 0$$

ε の1次に注目すると.

$$\frac{d^2x_1}{d\tau^2} + (\omega_1 + \omega_1)\frac{d^2x_0}{d\tau^2} + x_1 + x_0^3 = 0$$

$$\therefore \frac{d^2x_1}{d\tau^2} + x_1 = -2\omega_1\frac{d^2x_0}{d\tau^2} - x_0^3 \quad //$$

(iii) (A) の微分方程式の解は

$$x_0(\tau) = Ae^{i\tau} + Be^{-i\tau}$$

$$\therefore x_0(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

初期条件より

$$\left.\begin{aligned} A + B &= a \\ A - B &= 0 \end{aligned}\right\} \rightarrow A = B = \frac{a}{2}$$

$$x_0(t) = \frac{a}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) = a\cos\omega t \quad //$$

(iv) (B) に (A) の解を代入する。

$$\frac{d^2 x}{d\tau^2} = -a\cos\tau$$

$$\frac{d^2 x_1}{d\tau^2} + x_1 = +2\omega_1 a\cos\tau - a^3\cos^3\tau$$

同次解 $\frac{d^2 x_1}{d\tau^2} + x_1 = 0$ を求めると. $x_1 = A\underbrace{e^{i\tau}}_{=y_1} + B\underbrace{e^{-i\tau}}_{=y_2}$

ここで特殊解を求める.

$$(\text{右辺}) = 2\omega_1 a\cos\tau - a^3\cos\tau\cos^2\tau$$

$$= 2\omega_1 a\cos\tau - a^3\cos\tau\,\frac{1-\cos 2\tau}{2}$$

$$= 2\omega_1 a\cos\tau - \frac{1}{2}a^3\cos\tau + \frac{1}{2}a^3\cos\tau\cos 2\tau$$

$$\left(\begin{array}{l} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{ より} \\ \cos\tau\cos 2\tau = \frac{1}{2}(\cos 3\tau + \cos\tau) \end{array}\right)$$

$$= 2\omega_1 a\cos\tau - \frac{1}{2}a^3\cos\tau + \frac{1}{4}a^3\cos 3\tau + \frac{1}{4}a^3\cos\tau.$$

$$= 2\omega_1 a\cos\tau - \frac{1}{4}a^3\cos\tau + \frac{1}{4}a^3\cos 3\tau$$

となる.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} =$$