数学A

(1) $\cos(x-y) + i\sin(x-y)$

$$= e^{i(x-y)}$$

$$= e^{ix} \cdot e^{-iy}$$

$$= (\cos x + i\sin x)(\cos(-y) + i\sin(-y))$$

$$= (\cos x + i\sin x)(\cos y - i\sin y)$$

$$= \cos x\cos y + \sin x\sin y$$
$$+ i(\sin x\cos y - \cos x\sin y)$$

ゆえに、$\sin(x-y) = \sin x\cos y - \cos x\sin y$ //

(2) (i) $\frac{1}{1-x} = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots$

$f(x)$ (|||)

$$= 1 + x + x^2 + x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} x^n \quad //$$

(ii) $\log(1+x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots$

$f(x)$ (|||)

$$= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$$

(3) (i) $\int_0^{\infty} xe^{-x}dx = [-xe^{-x}]_0^{\infty} + \int_0^{\infty} e^{-x}dx$

$$= 0 + [-e^{-x}]_0^{\infty}$$

$$= 1 \quad //$$

(ii) $x = \frac{1}{t}$ とおくと $dx = -\frac{1}{t^2}dt$

x	$1 \to \infty$
t	$1 \to 0$

$$\int_1^0 \frac{-1/t^2\,dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}\right)^2-1}} = \int_0^1 \frac{dt}{\sqrt{1-t^2}}$$

さらに $t=\sin\theta$ とおくと. $dt=\cos\theta\,d\theta$

t	$0 \to 1$
θ	$0 \to \pi/2$

$$= \int_0^{\pi/2} \frac{\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$$

($\sqrt{1-\sin^2\theta} = \cos\theta$)

(4) $A=\begin{pmatrix}2 & 2\\ 2 & -1\end{pmatrix}$

(i) Aの固有値は. $\det(A-\lambda E)=0$ より.

$$A-\lambda E = \begin{pmatrix}2-\lambda & 2\\ 2 & -1-\lambda\end{pmatrix}$$

$$\begin{aligned}\det(A-\lambda E) &= -(2-\lambda)(1+\lambda)-4\\ &= (\lambda-2)(\lambda+1)-4\\ &= \lambda^2-\lambda-2-4\\ &= \lambda^2-\lambda-6\\ &= (\lambda-3)(\lambda+2)=0\end{aligned}$$

以上より $\lambda=3,-2$.

固有値ベクトルを求める。以下、tは任意。

$\lambda=3$ のとき

$$\begin{pmatrix}-1 & 2\\ 2 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=0$$

$$\left.\begin{aligned}-x+2y&=0\\ 2x-4y&=0\end{aligned}\right\} \to x=2y \qquad \therefore\ t\begin{pmatrix}2\\ 1\end{pmatrix}$$

$\lambda=-2$ のとき

$$\begin{pmatrix}4 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=0$$

$$\left.\begin{array}{l} 4x + 2y = 0 \\ 2x + y = 0 \end{array}\right\} \to y = -2x$$

$$\therefore\ t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$ //

(ii) Aの直交行列をPとすると.

$$P^{-1}AP = D$$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

また. $P = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ である。

(iii)

$$Q = 2x_1^2 + 4x_1x_2 - x_2^2$$

$$= (x_1\ x_2)\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \boldsymbol{x}^T A \boldsymbol{x}$$

$$= \boldsymbol{x}^T P D P^{-1} \boldsymbol{x}$$

$$= \boldsymbol{x}^T P D P^T \boldsymbol{x}$$

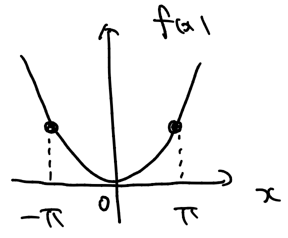
$$= (P^T\boldsymbol{x})^T D P^T\boldsymbol{x}$$

$$= \boldsymbol{y}^T D \boldsymbol{y}$$

$$= \frac{3}{5}(2x_1 + x_2)^2 - \frac{2}{5}(x_1 - 2x_2)^2$$ //

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix}$$

(5) (i)



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

n=0のとき

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\, dx$$

$$= \frac{2}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2}{3}\pi^2$$ //

n≠0のとき

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

~~$= \frac{2}{\pi}\left[\frac{x^2}{n}\sin nx\right]_0^{\pi}$~~

$$- \frac{2}{\pi}\int_0^{\pi} 2\frac{x}{n}\sin nx\, dx.$$

$$= -\frac{4}{n\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{4}{n\pi}\left[\frac{x}{n}\cos nx\right]_0^{\pi} - \frac{4}{n^2\pi}\int_0^{\pi}\cos nx \, dx$$

$$= \frac{4}{n^2\pi}\cdot\pi(-1)^n - \frac{4}{n^2\pi}\cancel{\left[\frac{1}{n}\sin nx\right]_0^{\pi}}$$

$$= \frac{4}{n^2}(-1)^n$$

(偶) (奇)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx = 0$$

以上より. $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$

(ii) $S = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n = 0$$

$$\therefore \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

ゆえに. $S = -\frac{\pi^2}{12}$ //