数学 B

(1) (i)
$$\begin{cases} \xi = x + iy \\ \xi^* = x - iy \end{cases}$$ であるから.

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \xi^*}{\partial x}\frac{\partial}{\partial \xi^*}$$

$$= \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi^*}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \xi^*}{\partial y}\frac{\partial}{\partial \xi^*}$$

$$= i\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \xi^*}$$

であるから.

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi^*}\right)^2 - \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi^*}\right)^2.$$

$$= \left(\cancel{\frac{\partial^2}{\partial \xi^2}} + 2\frac{\partial^2}{\partial \xi \partial \xi^*} + \cancel{\frac{\partial^2}{\partial \xi^{*2}}}\right)$$

$$- \left(\cancel{\frac{\partial^2}{\partial \xi^2}} - 2\frac{\partial^2}{\partial \xi \partial \xi^*} + \cancel{\frac{\partial^2}{\partial \xi^*}}\right)$$

$$= 4\frac{\partial^2}{\partial \xi \partial \xi^*}$$

∎

(ii)
$$\frac{\partial}{\partial \xi} = \left(\frac{\partial r}{\partial \xi}\right)_{\xi^*}\frac{\partial}{\partial r} + \left(\frac{\partial \varphi}{\partial \xi}\right)_{\xi^*}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial \xi^*} = \left(\frac{\partial r}{\partial \xi^*}\right)_{\xi}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial \xi^*}\frac{\partial}{\partial \varphi}$$

$$\left(\frac{\partial r}{\partial \xi}\right)_{\xi^*} = e^{i\varphi},\quad \left(\frac{\partial \varphi}{\partial \xi}\right)_{\xi^*} = \frac{1}{r i e^{i\varphi}},\quad \left(\frac{\partial r}{\partial \xi^*}\right)_{\xi} = e^{i\varphi},\quad \left(\frac{\partial \varphi}{\partial \xi^*}\right)_{\xi} = \frac{1}{-ir e^{i\varphi}}$$

$$\frac{\partial}{\partial \xi} = e^{-i\varphi}\frac{\partial}{\partial r} - \frac{i}{re^{i\varphi}}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial \xi^*} = e^{i\varphi}\frac{\partial}{\partial r} + \frac{i}{re^{-i\varphi}}\frac{\partial}{\partial \varphi}$$

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(2) (i) $u(r,\varphi) = R(r)\Phi(\varphi)$ とおくと.

$$\Phi(\varphi)\frac{\partial^2 R}{\partial r^2} + \frac{\Phi}{r}\frac{\partial R}{\partial r} + \frac{R(r)}{r^2}\frac{\partial^2}{\partial \varphi^2}\Phi = 0.$$

$$\Phi\frac{\partial^2 R}{\partial r^2} + \frac{\Phi}{r}\frac{\partial R}{\partial r} = -\frac{R}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2}$$

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} = -\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} = \alpha.$$

を満たすので

$$\begin{cases} \dfrac{\partial^2 R}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial R}{\partial r} - \dfrac{\alpha}{r^2}R = 0 \\ \dfrac{\partial^2 \Phi}{\partial \varphi^2} + \alpha\Phi = 0 \end{cases}$$

となる。

(ii) (1)は. $\Phi = Ae^{-i\sqrt{\alpha}\varphi} + Be^{i\sqrt{\alpha}\varphi}$ となる。

また. $\Phi(2\pi+\varphi) = \Phi(\varphi)$ より

$$\Phi = A\underbrace{\left(e^{-i2\pi\sqrt{\alpha}\varphi}\right)}_{=1}e^{-i\sqrt{\alpha}\varphi} + B\underbrace{\left(e^{i2\pi\sqrt{\alpha}\varphi}\right)}_{=1}e^{i\sqrt{\alpha}\varphi}$$

なので $\sqrt{a} = m$ (mは整数)

よって $a = m^2$.

これを $m>0$ としても同じ。$m>0$ にすれば重複しない.

(iii) $R = r^{\lambda}$ を仮定すると

$$\lambda(\lambda-1)r^{\lambda-2} + \frac{1}{r}\cdot\lambda r^{\lambda-1} - \frac{1}{r^2}\alpha\cdot r^{\lambda} = 0$$

$$\lambda^2 - \lambda + \lambda - \alpha = 0$$

$$\lambda^2 = \alpha$$

$$\lambda = \pm m.$$

$$\therefore R = r^{\pm m}$$

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(iv) $u(\infty,\varphi) = 0$ より. $R = r^{-m}$

$$u(a,\varphi) = \sum_m a^{-m}\left(A_m e^{-im\varphi} + B_m e^{im\varphi}\right)$$

$$= \sum_m a^{-m}A_m(\cos m\varphi - i\sin m\varphi) + a^{-m}B_m(\cos m\varphi + i\sin m\varphi)$$

$$= \sum_m a^{-m}(A_m + B_m)\cos m\varphi + i a^{-m}(B_m - A_m)\sin m\varphi.$$

$$= \underbrace{\boxed{a^{-1}(A_1+B_1)}}_{=3}\cos\varphi + \underbrace{\boxed{i a^{-1}(B_1 - A_1)}}_{=0}\sin\varphi$$

$$+ \underbrace{\boxed{a^{-2}(A_2+B_2)}}_{=0}\cos 2\varphi + \underbrace{\boxed{i a^{-2}(B_2 - A_2)}}_{=5}\sin 2\varphi$$

$$\left[+ \underbrace{\boxed{a^{-m}(A_m + A_m)}}_{=0}\cos m\varphi + \underbrace{\boxed{i a^{-m}(B_m - A_m)}}_{=0}\sin m\varphi \right]$$

$m > 3$

$$2A_1 a^{-1} = 3 \qquad 2i a^{-2} B_2 = 5$$

$$A_1 = \frac{3a}{2} = B_1 \qquad B_2 = \frac{5a^2}{2i} = -A_2$$

以上より.
$$u(r,\varphi) = r^{-1}\left\{\frac{3a}{\boxed{2}}\underbrace{\boxed{\left(e^{-i\varphi} + e^{i\varphi}\right)}}_{=\cos\varphi}\right\} + r^{-2}\left\{\frac{5a^2}{\boxed{2i}}\underbrace{\boxed{\left(e^{i\varphi} - e^{-i\varphi}\right)}}_{=\sin 2\varphi}\right\}$$

$$= \frac{3a}{r}\cos\varphi + \frac{5a^2}{r^2}\sin 2\varphi$$

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