数学A

(1) $\cos n\theta + i\sin n\theta = e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$

(2) $\dfrac{1}{\sqrt{1+x}} = 1 - \dfrac{1}{2}x + \dfrac{3}{8}x^2 - \dfrac{15}{48}x^3$

(3) (i)

$$\int_0^{\pi} e^x \sin x\,dx = \cancel{\left[e^x \sin x\right]_0^{\pi}} - \int_0^{\pi} e^x \cos x\,dx$$

$$= -\left[e^x \cos x\right]_0^{\pi} - \int_0^{\pi} e^x \sin x\,dx$$

$$= -(-e^{\pi} - 1) - \int_0^{\pi} e^x \sin x\,dx$$

$$\therefore \int_0^{\pi} e^x \sin x\,dx = \frac{e^{\pi}+1}{2} \quad /\!/$$

(ii) $x = \cos\theta$ とおく

$dx = -\sin\theta\,d\theta$

x	$0 \to 1$
θ	$\frac{\pi}{2} \to 0$

$$\int_0^1 x\sqrt{1-x^2}\,dx = \int_{\frac{\pi}{2}}^{0} -\cos\theta \sin^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta\,d\theta$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \int_0^{\frac{\pi}{2}} \underbrace{\frac{1-\cos 2\theta}{2}\cos\theta}_{※}\,d\theta$$

※の部分

$\frac{1}{2}\cos\theta - \frac{1}{2}\cos 2\theta\cos\theta$

$= \frac{1}{2}\cos\theta - \frac{1}{4}(\cos 3\theta + \cos\theta)$

$= \frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta \quad d\theta$$

$$= \left[\frac{1}{4}\sin\theta\right]_0^{\frac{\pi}{2}} - \left[\frac{1}{12}\sin 3\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3} \;//$$

(4) $A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$

(i) 固有値方程式 $\det(A - \lambda E) = 0$ より.

$$(-1-\lambda)^2 - 9 = 0$$
$$\lambda^2 + 2\lambda + 1 - 9 = 0$$
$$\lambda^2 + 2\lambda - 8 = 0$$
$$(\lambda+4)(\lambda-2) = 0$$

$$\lambda = 2, -4 \;//$$

固有値は $2, -4$ //

$\lambda = 2$ のとき.

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left.\begin{array}{l} -3x + 3y = 0 \\ 3x - 3y = 0 \end{array}\right\} \to x = y.$$

固有値ベクトルは. $t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
t: 任意

$\lambda = -4$ のとき

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left.\begin{array}{l} 3x + 3y \\ 3x + 3y \end{array}\right\} \to x = -y$$

固有値ベクトルは
t: 任意

$t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ //

(ii) $P' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ とすると. $P'^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$D = P'^{-1} A P'$

$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} 2 & 2 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$ //

また.
直交行列 $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ //

(iii) $Q = -x_1^2 + 6x_1x_2 + x_2^2$

$A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$ $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

とすると.

$Q = \mathbf{x}^T A \mathbf{x}$

$= \mathbf{x}^T P D P^{-1} \mathbf{x}$

$= (P^T \mathbf{x})^T D (\underline{\underline{P^{-1}}} \mathbf{x})$

$= (P^T \mathbf{x})^T D (\underline{\underline{P^{T}}} \mathbf{x})$) 直交行列なので

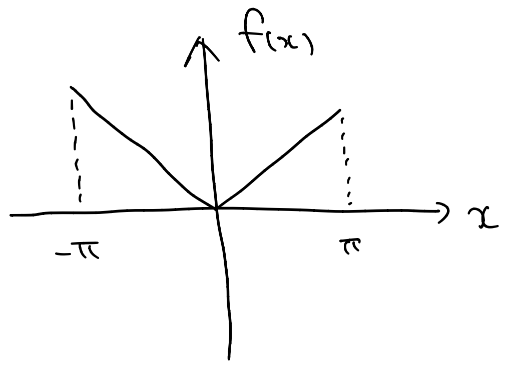
$= \mathbf{y}^T D \mathbf{y}$ $\mathbf{y} \equiv P^T \mathbf{x}$ とすると.

$= (y_1, y_2)\begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$= 2y_1^2 - 4y_2^2$

ただし. $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 \\ \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2 \end{pmatrix}$

(5) (i)



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cos nx \, dx$$

$n=0$ のとき

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx = \frac{\cancel{2}}{\pi}\left[\frac{1}{\cancel{2}}x^2\right]_0^{\pi} = \pi$$ //

$n \neq 0$ のとき

$$\int_0^{\pi} x \cos nx \, dx = \left[\frac{x}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi} \frac{1}{n}\sin nx \, dx$$

$$= 0 + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi}$$

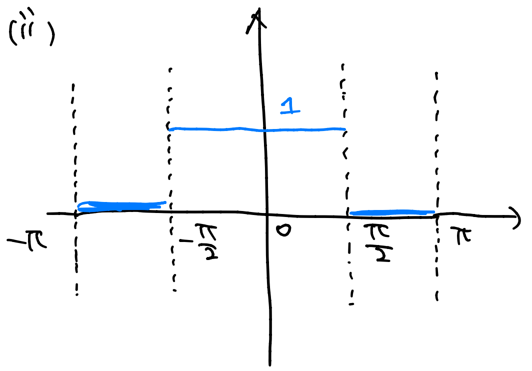
$$= \frac{1}{n^2}\{(-1)^n - 1\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx \, dx = 0.$$

$$\therefore \quad f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{n^2}\{(-1)^n - 1\} \cos nx$$

//

(ii)



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} \cos nx \, dx$$

$$= \begin{cases} n=0 \text{ のとき} \\ \dfrac{2}{\pi}\displaystyle\int_{0}^{\frac{\pi}{2}} dx = \dfrac{2}{\pi}\cdot\dfrac{\pi}{2} = 1 \\ \\ n\neq 0 \text{ のとき} \\ \dfrac{2}{\pi}\left[\dfrac{1}{n}\sin nx\right]_{0}^{\pi/2} = \dfrac{2}{n\pi}\sin\dfrac{n\pi}{2} \end{cases}$$

偶

$$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx = 0$$

奇

$$\therefore\ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx$$

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