数学B

(1)

$$l = \int_0^T \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}\, ds$$

$$\begin{cases} \dfrac{dx}{ds} = -a \sin s \\ \dfrac{dy}{ds} = a \cos s \\ \dfrac{dz}{ds} = b \end{cases}$$

以上より

$$l = \int_0^T \sqrt{a^2 \sin^2 s + a^2 \cos^2 s + b^2}\, ds$$

$$= \int_0^T \sqrt{a^2 + b^2}\, ds$$

$$= \sqrt{a^2 + b^2}\, T \quad //$$

(2) (i)

$$\frac{\partial u}{\partial x} = \frac{-2yz}{(x^2+y^2)^2} + \frac{8x^2yz}{(x^2+y^2)^3}$$

$$= \frac{-2yz(x^2+y^2) + 8x^2yz}{(x^2+y^2)^3}$$

$$= \frac{6x^2yz - 2y^3z}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial y} = \frac{-2xz}{(x^2+y^2)^2} + \frac{8xy^2z}{(x^2+y^2)^3}$$

$$= \frac{-2xz(x^2+y^2) + 8xy^2z}{(x^2+y^2)^3}$$

$$= \frac{6xy^2z - 2xy^2z}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial z} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xz}{(x^2+y^2)^2} - \frac{4x(x^2-y^2)z}{(x^2+y^2)^3}$$

$$= \frac{2xz(x^2+y^2)}{(x^2+y^2)^3} - \frac{4xz(x^2-y^2)}{(x^2+y^2)^3} = \frac{-2x^3z+6xzy^2}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{-2yz}{(x^2+y^2)^2} - \frac{4y(x^2-y^2)z}{(x^2+y^2)^3}$$

$$= \frac{-2yz(x^2+y^2)}{(x^2+y^2)^3} - \frac{4yz(x^2-y^2)}{(x^2+y^2)^3} = \frac{-6x^2yz+2y^3z}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial z} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial w}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial w}{\partial y} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial w}{\partial z} = 0$$

(ii)

$$\mathrm{div}\, \boldsymbol{v} = \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}\right)$$ ∵ 詳しい成分は(i)を参照

$$\mathrm{rot}\, \boldsymbol{v} = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ u & v & w \\ \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \end{vmatrix}$$

$$= (\partial_y w - \partial_z v)\boldsymbol{e}_x + (\partial_z u - \partial_x w)\boldsymbol{e}_y + (\partial_x v - \partial_y u)\boldsymbol{e}_z$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}\boldsymbol{e}_x + \left[\frac{-2xy}{(x^2+y^2)^2} - \frac{-2xy}{(x^2+y^2)^2}\right]\boldsymbol{e}_y$$

$$+ \left[\frac{-2x^3z+6xzy^2}{(x^2+y^2)^3} - \frac{6xy^2z-2xy^2z}{(x^2+y^2)^3}\right]\boldsymbol{e}_z$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}\boldsymbol{e}_x$$ $\boldsymbol{e}_x$ : $x$ の単位ベクトル.

(3) (i) $d\tau = \omega\, dt$ なので

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \omega \frac{dx}{d\tau}$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \omega \frac{d}{dt}\left(\frac{dx}{d\tau}\right) = \omega \frac{d\tau}{dt}\frac{d}{d\tau}\left(\frac{dx}{d\tau}\right) = \omega^2 \frac{d^2x}{d\tau^2}$$

以上より

$$\omega^2 \frac{d^2x}{d\tau^2} + x + \varepsilon x^3 = 0 \quad \text{—— } (*)$$

(ii)

$$\left.\begin{aligned} x &= x_0 + \varepsilon x_1 + \cdots \\ \omega &= 1 + \varepsilon \omega_1 + \cdots \end{aligned}\right\}$$ を $(*)$ に代入する.

$$(1+\varepsilon\omega_1+\cdots)^2\left(\frac{d^2x_0}{d\tau^2} + \varepsilon\frac{d^2x_1}{d\tau^2} + \cdots\right)$$
$$+(x_0+\varepsilon x_1+\cdots) + \varepsilon(x_0+\varepsilon x_1+\cdots)^3 = 0.$$

$\varepsilon$ の0次に注目すると,

$$\frac{d^2x_0}{d\tau^2} + x_0 = 0$$

$\varepsilon$ の1次に注目すると.

$$\frac{d^2x_1}{d\tau^2} + (\omega_1 + \omega_1)\frac{d^2x_0}{d\tau^2} + x_1 + x_0^3 = 0$$

$$\therefore \frac{d^2x_1}{d\tau^2} + x_1 = -2\omega_1\frac{d^2x_0}{d\tau^2} - x_0^3 \quad //$$

(iii) (A)の線形同次方程式の解は

$$x_0(\tau) = Ae^{i\tau} + Be^{-i\tau}$$
$$\therefore x_0(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

初期条件より
$$\left.\begin{aligned} A + B &= a \\ A - B &= 0 \end{aligned}\right\} \to A = B = \frac{a}{2}$$

$$x_0(t) = \frac{a}{2} e^{i\omega t} + e^{-i\omega t} = a\cos\omega t \quad //$$

(iv)　(B)に(A)の解を代入する。

$$\frac{d^2x}{d\tau^2} = -a\cos\tau$$

$$\frac{d^2x_1}{d\tau^2} + x_1 = +2\omega_1 a\cos\tau - a^3\cos^3\tau$$

同次解　$\frac{d^2x_1}{d\tau^2} + x_1 = 0$　を求めると．　$x_1 = A\underbrace{e^{i\tau}}_{=y_1} + B\underbrace{e^{-i\tau}}_{=y_2}$

ここで特殊解を求める．

$$(右辺) = 2\omega_1 a\cos\tau - a^3\cos\tau\cos^2\tau$$

$$= 2\omega_1 a\cos\tau - a^3\cos\tau\,\frac{1-\cos 2\tau}{2}$$

$$= 2\omega_1 a\cos\tau - \frac{1}{2}a^3\cos\tau + \frac{1}{2}a^3\cos\tau\cos 2\tau$$

$$\left(\begin{aligned} &\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{ より} \\ &\cos\tau\cos 2\tau = \frac{1}{2}(\cos 3\tau + \cos\tau) \end{aligned}\right)$$

$$= 2\omega_1 a\cos\tau - \frac{1}{2}a^3\cos\tau + \frac{1}{4}a^3\cos 3\tau + \frac{1}{4}a^3\cos\tau.$$

$$= 2\omega_1 a\cos\tau - \frac{1}{4}a^3\cos\tau + \frac{1}{4}a^3\cos 3\tau$$

となる．

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} =$$