数学 A

問1

(1) (i)

$$\frac{dy}{dx} + xy = x$$

$$\frac{dy}{dx} = x(1-y)$$

$$\frac{1}{1-y}\,dy = x\,dx.$$

$$\log(1-y) = \frac{1}{2}x^2 + \mathrm{Const}$$

$$1-y = Ae^{\frac{1}{2}x^2} \quad (A:\mathrm{Const})$$

$$\therefore\ y = Ae^{\frac{1}{2}x^2} + 1$$

(ii) 非同次方程式ではなく 同次方程式:

$$\frac{dy}{dx} + y = 0 \quad \text{—— ①}$$

を求める。①の一般解は $y = Ae^{-x}$

$$\boxed{\begin{aligned}&\frac{dy}{dx} = -y\\&\frac{1}{y}\,dy = -dx.\\&\log y = -x + \mathrm{Const}\\&y = Ae^{-x}\end{aligned}}$$

ここで $y = ax + b$ とおいて $\frac{dy}{dx} + y = x$ を解いてみる.

$$a + ax + b = x.$$

$$(a-1)x + a + b = 0.$$

$$\left.\begin{aligned}a-1&=0\\a+b&=0.\end{aligned}\right\} \rightarrow \begin{aligned}a&=1\\b&=-1\end{aligned}$$

以上より. $\frac{dy}{dx} + y = x$ の一般解は. $y = Ae^{-x} + x - 1$

(iii) $z = y^{1-n}$ より $\frac{dy}{dz} = \frac{1}{1-n}y^n$ が言える.

$$\boxed{\begin{aligned}&dz = (1-n)y^{-n}\,dy.\\&\frac{1}{1-n}y^n = \frac{dy}{dz}\end{aligned}}$$

よって.

$$y^{-n}\frac{dy}{dz}\cdot\frac{dz}{dx} + f_1(x)\,y^{1-n} = f_2(x)$$

$$\cancel{y^{-n}}\,\frac{1}{1-n}\,\cancel{y^n}\,\frac{dz}{dx} + f_1(x)\,z = f_2(x).$$

$$\frac{1}{1-n}\frac{dz}{dx} + f_1(x)\,z = f_2(x).$$ //

(iv)

$$\frac{dy}{dx} + xy = y^2x.$$

$$\frac{dy}{dx} = x(y^2-y)$$

$$\frac{1}{y^2-y}\,dy = x\,dx.$$

$$\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = x$$

$$\log(y-1) - \log y = \frac{1}{2}x^2 + \mathrm{Const}$$

$$\log\frac{y-1}{y} = \frac{1}{2}x^2 + \mathrm{Const}$$

$$\frac{y-1}{y} = Ae^{-\frac{1}{2}x^2} \quad (A:\mathrm{const})$$

$$1 - \frac{1}{y} = Ae^{-\frac{1}{2}x^2}$$

$$\frac{1}{y} = Ae^{-\frac{1}{2}x^2} + 1 \qquad \therefore\ y = \frac{1}{e^{-\frac{1}{2}x^2} + 1}$$ //

(2) (i) 右辺は、$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$ であるから、

$$\boldsymbol{\lambda}\cdot\nabla\phi = \lambda_x\frac{\partial\phi}{\partial x} + \lambda_y\frac{\partial\phi}{\partial y} + \lambda_z\frac{\partial\phi}{\partial z}$$

左辺は、

$$\frac{\partial\phi}{\partial\lambda} = \lim_{h\to 0}\frac{\phi(x+h\lambda_x, y+h\lambda_y, z+h\lambda_z) - \phi(x, y+h\lambda_y, z+h\lambda_z)}{h}$$

$$+ \frac{\phi(x, y+h\lambda_y, z+h\lambda_z) - \phi(x, y, z+h\lambda_z)}{h}$$

$$+ \frac{\phi(x, y, z+h\lambda_z) - \phi(x, y, z)}{h}$$

$$= \lim_{h\to 0}\lambda_x\frac{\phi(x+h\lambda_x, y+h\lambda_y, z\lambda_z) - \phi(x, y+h\lambda_y, z+h\lambda_z)}{h\lambda_x}$$

$$+ \cdots$$

$$= \lambda_x\frac{\partial\phi}{\partial x} + \lambda_y\frac{\partial\phi}{\partial y} + \lambda_z\frac{\partial\phi}{\partial z} \quad //$$

(ii) (i) より $\frac{\partial\phi}{\partial n} = \mathbf{n}\cdot\nabla\phi$ が言える。

両辺に $\mathbf{n}$ をかける。

$$\frac{\partial\phi}{\partial n}\mathbf{n} = \mathbf{n}(\mathbf{n}\cdot\nabla\phi)$$

$$=$$

$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\psi = 0$

→ 保存力である
どこで使った?

(iii) 右辺について、

$$\left[\frac{1}{2}\nabla v^2\right]_i = \frac{1}{2}\partial_i v_j v_j$$

$$= \frac{1}{2}\left[v_j\partial_i v_j + v_j\partial_i v_j\right] = v_j\partial_i v_j$$

$$[\boldsymbol{v}\times\mathrm{rot}\,\boldsymbol{v}]_i = \varepsilon_{ijk}v_j[\mathrm{rot}\,\boldsymbol{v}]_k$$

$$= \varepsilon_{ijk}v_j\varepsilon_{klm}\partial_l v_m$$

$$= \varepsilon_{ijk}\varepsilon_{klm}v_j\partial_l v_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})v_j\partial_l v_m$$

$$= \delta_{il}\delta_{jm}v_j\partial_l v_m - \delta_{im}\delta_{jl}v_j\partial_l v_m$$

$$= v_j\partial_i v_j - v_j\partial_j v_i$$

であるから、

$$(右辺) = \left[\frac{1}{2}\nabla v^2 - \boldsymbol{v}\times\mathrm{rot}\,\boldsymbol{v}\right]_i = \cancel{v_j\partial_i v_j - v_j\partial_i v_j} + v_j\partial_j v_i$$

$$(\text{左辺}) = [(v \cdot \nabla) v]_i = v_j \partial_j v_i = (\text{右辺}) \quad \blacksquare$$

(iv)

$$\frac{\partial}{\partial l}\left(\frac{1}{2}v^2 + \psi\right) = l \cdot \left(\frac{1}{2}\nabla v^2 + \nabla\psi\right)$$

$$= l \cdot \left(\cancel{(v \cdot \nabla) v} + v \times \mathrm{rot}\, v + \cancel{\nabla\psi}\right)$$

$$= l \cdot (v \times \mathrm{rot}\, v)$$

$$=$$