数学B

(1) $\displaystyle\sum_{n=0}^{\infty}\frac{a^n}{n!} = \exp(a)$ なので $\displaystyle\sum_{n=1}^{\infty}\frac{a^n}{n!} = \exp(a) - 1$ に収束する。

(2)
$$\int_0^1\int_0^1 y^2 e^{xy}\,dx\,dy = \int_0^1 (ye^y - y)\,dy$$
$$= \underbrace{\int_0^1 ye^y\,dy}_{=1} - \underbrace{\int_0^1 y\,dy}_{=\left[\frac{1}{2}y^2\right]_0^1 = \frac{1}{2}}$$
$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$\boxed{\begin{aligned}\int_0^1 ye^y\,dy &= \left[ye^y\right]_0^1 - \int_0^1 e^y\,dy\\ &= e - \left[e^y\right]_0^1\\ &= e - e + 1\end{aligned}}$$

(3)
$$y = \frac{e^x - e^{-x}}{2}$$
$$2y = e^x - \frac{1}{e^x}$$
$$2ye^x = e^{2x} - 1$$
$$0 = e^{2x} - 2ye^x - 1$$

$\lambda = e^x\ (>0)$ とおくと
$$0 = \lambda^2 - 2y\lambda - 1$$
$$\lambda = y + \sqrt{y^2+1} \quad (\lambda > 0 \text{ より})$$
$$e^x = y + \sqrt{y^2+1}$$
$$x = \log(y + \sqrt{y^2+1})$$

以上より、$\sinh^{-1}x = \log(x + \sqrt{x^2+1})$

(4)
$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{pmatrix}$$

(i) 行列式
$$\det|A| = (-1+0+0) - (6+2+0)$$
$$= -1 - 8 = -9$$

(ii) 逆行列.
$$\tilde{A}_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 1\cdot(-1-2) = -3$$
$$\tilde{A}_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = -1\cdot(0-2) = 2$$
$$\tilde{A}_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} = 1\cdot(0-2) = -2$$

$$\tilde{A}_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} = -1 \cdot (-6) = 6$$

$$\tilde{A}_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = 1 \cdot (-1-6) = -7$$

$$\tilde{A}_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = -1 \cdot (2-0) = -2$$

$$\tilde{A}_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 1 \cdot (0-3) = -3$$

$$\tilde{A}_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = -1 \cdot (1-0) = -1$$

$$\tilde{A}_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot (1-0) = 1$$

ゆえに $\tilde{A} = \begin{bmatrix} -3 & 2 & -2 \\ 6 & -7 & -2 \\ -3 & -1 & 1 \end{bmatrix}$

逆行列 $A^{-1} = \frac{1}{|A|} \tilde{A}^T$

$$= -\frac{1}{9} \begin{bmatrix} -3 & 6 & -3 \\ 2 & -7 & -1 \\ -2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{9} & \frac{7}{9} & \frac{1}{9} \\ \frac{2}{9} & \frac{2}{9} & -\frac{1}{9} \end{bmatrix}$$

(iii) Aの固有値を求める。

固有値方程式 $\det(A - \lambda E) = 0$

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & 1 \\ 2 & 2 & -1-\lambda \end{pmatrix}$$

$$\det(A - \lambda E) = -(1-\lambda)^2(1+\lambda) - 6(1-\lambda) - 2(1-\lambda)$$

$$= (\lambda - 1)\left[(1-\lambda)(1+\lambda) + 6 + 2\right]$$

$$= (\lambda-1)(1-\lambda^2+8)$$

$$= (\lambda-1)(\lambda-3)(\lambda+3) = 0$$

固有値は $\lambda = -3, 1, 3$ //

(iv) 固有値ベクトルを求める.

(a) $\lambda = 3$ のとき.

$$\begin{bmatrix} -2 & 0 & 3 \\ 0 & -2 & 1 \\ 2 & 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\left.\begin{array}{l} -2x + 3z = 0 \\ -2y + z = 0 \\ 2x + 2y - 4z = 0 \end{array}\right\} \rightarrow \begin{array}{l} x = \frac{3}{2}z \\ y = \frac{1}{2}z \end{array}$$

∴ 固有値ベクトルは $t\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$
(t：任意)

(b) $\lambda = 1$ のとき

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \\ 2 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\left.\begin{array}{l} 3z = 0 \\ z = 0 \\ 2x + 2y - 2z = 0 \end{array}\right\} \rightarrow \begin{array}{l} x = -y \\ z = 0 \end{array}$$

∴ 固有値ベクトルは $t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
(t：任意)

(c) $\lambda = -3$ のとき.

$$\begin{bmatrix} 4 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\left.\begin{array}{l} 4x + 3z = 0 \\ 4y + z = 0 \\ 2x + 2y + 2z = 0 \end{array}\right\} \rightarrow \begin{array}{l} x = -\frac{3}{4}z \\ y = -\frac{1}{4}z \end{array}$$

∴ 固有値ベクトルは $t\begin{bmatrix} -\frac{3}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$
(t：任意)

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(v)

$$A^3 = (PDP^{-1})^3$$
$$= PD\underline{P^{-1}P}D\underline{P^{-1}P}D\underline{P^{-1}P}DP^{-1}$$
$$= PD^3P^{-1}$$

$$D = \begin{bmatrix} -3 & & 0 \\ & 1 & \\ 0 & & 3 \end{bmatrix}$$

$$A^2 = PD^2P^{-1}$$

$$PD^3P^{-1} + aPD^2P^{-1} + bPDP^{-1} + cE = O.$$

$$P(D^3 + aD^2 + bD)P^{-1} + cE = O.$$

$$P\begin{bmatrix} -27+9a-3b & & 0 \\ & 1+a+b & \\ 0 & & 27+9a+3b \end{bmatrix}P^{-1} = \begin{bmatrix} -c & & 0 \\ & -c & \\ 0 & & -c \end{bmatrix}$$

$$\begin{bmatrix} -27+9a-3b & & \\ & 1+a+b & \\ & & 27+9a+3b \end{bmatrix} = -cP^{-1}EP = -cE$$

ゆえに次の連立方程式を得る。

$$\begin{cases} -27+9a-3b = -c \\ 1+a+b = -c \\ 27+9a+3b = -c \end{cases}$$

$$-27+\cancel{9a}-3b = 27+\cancel{9a}+3b.$$
$$-54 = 6b$$
$$-9 = b$$
$$-\cancel{27}+9a+\cancel{27} = 1+a-9$$
$$8a = -8$$
$$a = -1$$

$$\begin{array}{rl} & -27+9a-3b = -c \\ +) & 27+9a+3b = -c \\ \hline & 18a = -2c \end{array}$$

$$9a = -c \rightarrow c = +9$$

$$\therefore\ \underline{a = -1,\ b = -9,\ c = 9}$$ //

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(%i1) solve([-27+9·a-3·b=-c,1+a+b=-c,27+9·a+3·b=-c],[a,b,c]);
(%o1) [[a=-1,b=-9,c=9]]
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(vi) $\underline{\underline{A^6}} - 2A^5 - 6A^4 + 14A^3 - 25A^2 + 30A - 12E$

$= \underline{\underline{\cancel{A^5} + 9\cancel{A^4} - \cancel{9A^3}}} - 2\cancel{A^5} - \cancel{6}A^4 + \cancel{14}A^3 - \cancel{25}A^2 + \cancel{30}A - 12E$

$= -A^5 + 3A^4 + 5A^3 - 25A^2 + 30A - 12E$

$= -(\cancel{A^4} + \cancel{9A^3} - \cancel{9A^2}) + \cancel{3A^4} + \cancel{5}A^3 - \cancel{25}A^2 + 30A - 12E$

$= 2A^4 - 4A^3 - 16A^2 + 30A - 12E$

$= 2(\cancel{A^3} + 9A^2 - 9A) - \cancel{4A^3} - 16A^2 + 30A - 12E$ $\qquad A^3 = A^2 + 9A - 9$

$= -2A^3 + 2A^2 + 12A - 12E$

$= -2(\cancel{A^2} + 9A - 9) + \cancel{2A^2} + 12A - 12E$

$= -18A + 18E + 12A - 12E$

$= -6A + 6E$

$$= \begin{bmatrix} 0 & 0 & -18 \\ 0 & 0 & -6 \\ -12 & -12 & 12 \end{bmatrix}$$