数学 B

(1) (i) $\nabla r = \frac{\partial r}{\partial x}\mathbb{i} + \frac{\partial r}{\partial y}\mathbb{j} + \frac{\partial r}{\partial z}\mathbb{k}$

$$= \frac{1}{r}(x\mathbb{i} + y\mathbb{j} + z\mathbb{k})$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}(x\mathbb{i} + y\mathbb{j} + z\mathbb{k}) \text{ //}$$

(ii) $\nabla \cdot \mathbb{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$ //

(iii) $\nabla \times \mathbb{r} = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ x & y & z \\ \mathbb{i} & \mathbb{j} & \mathbb{k} \end{vmatrix} = \mathbb{0}$

(iv) $\nabla^2\left(\frac{1}{r}\right)$ を計算するにあたり $\nabla\left(\frac{1}{r}\right)$ をまず計算する。

$$\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial r}{\partial x}\mathbb{i} + \frac{\partial r}{\partial y}\mathbb{j} + \frac{\partial r}{\partial z}\mathbb{k}\right)\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\right)}_{= -\frac{1}{r^2}}$$

$$= -\frac{x}{r^3}\mathbb{i} - \frac{y}{r^3}\mathbb{j} - \frac{z}{r^3}\mathbb{k}$$

$$\left[\nabla\cdot\left(\nabla\left(\frac{1}{r}\right)\right)\right]_x = \frac{\partial x}{\partial x}\left(-\frac{1}{r^3}\right) + x\underbrace{\frac{\partial r}{\partial x}}_{\frac{x}{r}}\cdot\underbrace{\frac{\partial}{\partial r}\left(-\frac{1}{r^3}\right)}_{\frac{3}{r^4}}$$

x成分

$$= -\frac{1}{r^3} + \frac{3x^2}{r^5}$$

$$\therefore \nabla\cdot\left(\nabla\left(\frac{1}{r}\right)\right) = -\frac{3}{r^3} + \frac{3r^2}{r^5} = 0 \text{ //}$$

(2) (i) $x = r\cos\theta$, $y = r\sin\theta$ と変数変換する。

ヤコビアン $\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$ より

$$\int_0^\infty\int_0^{2\pi} \exp(-ar^2)\, r\, dr\, d\theta = \boxed{\int_0^\infty 2\pi r \exp(-ar^2)\, dr}$$

$$= \left[-\frac{\pi}{a}\exp(-ar^2)\right]_0^\infty$$

$$= \frac{\pi}{a}$$

よって. $I^2 = \frac{\pi}{a}$ より $I = \sqrt{\frac{\pi}{a}}$ //

(ii) $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} \exp(-t)\, dt$

ここで $t^{\frac{1}{2}} = s$ とおくと $dt/ds = 2s$

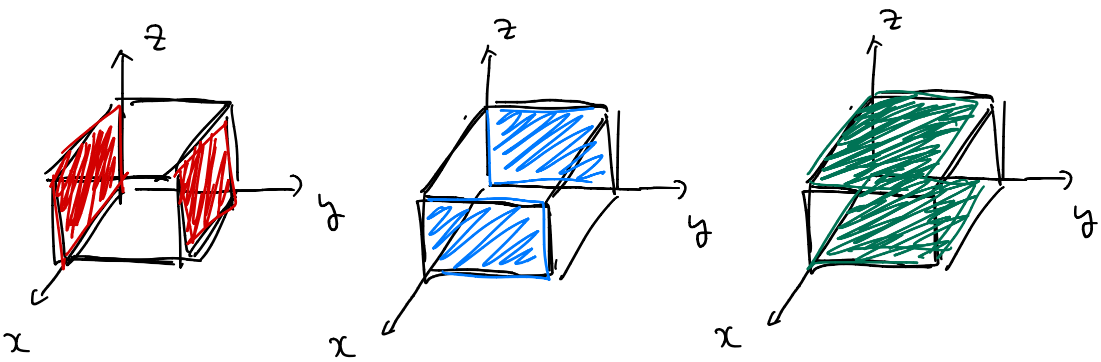
$$t^{-\frac{1}{2}} \exp(-t) = s^{-1} \exp(-s^2)$$

よって $t^{-\frac{1}{2}} \exp(-t)\, dt = 2 \exp(-s^2)\, ds$

$$\Gamma(\tfrac{1}{2}) = 2 \int_0^\infty \exp(-s^2)\, ds$$

$$= \cancel{2} \cdot \frac{\sqrt{\pi}}{\cancel{2}} = \pi$$

(3) (i) $\int_S \mathbb{A} \cdot \mathbb{n}\, dS = \int \mathbb{A} \cdot \mathbb{n}\, ds + \int \mathbb{A} \cdot \mathbb{n}\, ds + \int \mathbb{A} \cdot \mathbb{n}\, ds$



$$\int \mathbb{A} \cdot \mathbb{n}\, ds = -0 + \int_0^1 \int_0^1 5z\, dx\, dz = \frac{5}{2}$$

$$\int \mathbb{A} \cdot \mathbb{n}\, ds = -0 + \int_0^1 \int_0^1 y\, dy\, dz = \frac{1}{2}$$

$$\int \mathbb{A} \cdot \mathbb{n}\, ds = +0 - \int_0^1 \int_0^1 2y\, dx\, dy = -1$$

より $\int_S \mathbb{A} \cdot \mathbb{n}\, ds = \frac{5}{2} + \frac{1}{2} - 1 = 2$

(ii) $\nabla \cdot \mathbb{A} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(5yz) + \frac{\partial}{\partial z}(-2zy)$

$$= y + 5z - 2y = -y + 5z$$

$$\int_0^1 \int_0^1 \int_0^1 (-y + 5z)\, dx\, dy\, dz = \int_0^1 \left(-\frac{1}{2} + 5z\right) dz$$

$$= -\frac{1}{2} + \frac{5}{2} = 2$$ //