大問3

I (1) $\dfrac{\hat{L}_\pm}{\hbar}|L, m_L\rangle = \sqrt{L(L+1) - m_L(m_L \pm 1)}\,|L, m_L \pm 1\rangle$ より

$$\frac{\hat{L}_-}{\hbar}|1,1\rangle = \sqrt{2}\,|1,0\rangle$$

$$\frac{\hat{L}_-}{\hbar}|1,0\rangle = \sqrt{3}\,|1,-1\rangle$$

$$\frac{\hat{L}_-}{\hbar}|1,-1\rangle = 0$$

$$\frac{\hat{S}_-}{\hbar}\left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$\frac{\hat{S}_-}{\hbar}\left|\frac{1}{2},-\frac{1}{2}\right\rangle = 0$$

(2) $$\frac{\hat{J}_-}{\hbar}\left|\frac{3}{2},\frac{3}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle$$

$$\frac{\hat{J}_-}{\hbar}\left|\frac{3}{2},\frac{3}{2}\right\rangle = \frac{\hat{L}_- + \hat{S}_-}{\hbar}\left|1,1;\frac{1}{2},\frac{1}{2}\right\rangle$$

$$= \frac{\hat{L}_-}{\hbar}\left|1,1;\frac{1}{2},\frac{1}{2}\right\rangle + \frac{\hat{S}_-}{\hbar}\left|1,1;\frac{1}{2},\frac{1}{2}\right\rangle$$

$$= \sqrt{2}\left|1,0;\frac{1}{2},\frac{1}{2}\right\rangle + \left|1,1;\frac{1}{2},-\frac{1}{2}\right\rangle$$

以上より

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1,0;\frac{1}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|1,1;\frac{1}{2},-\frac{1}{2}\right\rangle$$

(3) $J = \frac{1}{2}$ のとき、$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \alpha\left|1,0;\frac{1}{2},\frac{1}{2}\right\rangle + \beta\left|1,1;\frac{1}{2},-\frac{1}{2}\right\rangle$ とすれば $\alpha^2+\beta^2 = 1$ を満たし、直交条件より

$$\left\langle\frac{3}{2},\frac{1}{2}\middle|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\,\alpha\,\underbrace{\left\langle 1,0;\frac{1}{2},\frac{1}{2}\middle|1,0;\frac{1}{2},\frac{1}{2}\right\rangle}_{=1} + \sqrt{\frac{1}{3}}\,\beta\,\underbrace{\left\langle 1,1;\frac{1}{2},-\frac{1}{2}\middle|1,1;\frac{1}{2},-\frac{1}{2}\right\rangle}_{=1} = 0$$

$$\therefore\ \sqrt{\frac{2}{3}}\,\alpha + \sqrt{\frac{1}{3}}\,\beta = 0$$

$$\begin{cases} \alpha^2+\beta^2=1 \\ \sqrt{\frac{2}{3}}\alpha+\sqrt{\frac{1}{3}}\beta=0 \end{cases} \rightarrow \quad \beta=-\sqrt{2}\alpha$$

$$\alpha^2+2\alpha^2=1$$

$$3\alpha^2=1$$

$$\alpha^2=\frac{1}{3}$$

$$\alpha=\sqrt{\frac{1}{3}} \qquad \beta=-\sqrt{\frac{2}{3}}$$

以上より

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle=\sqrt{\frac{1}{3}}\left|1,0;\frac{1}{2},\frac{1}{2}\right\rangle-\sqrt{\frac{2}{3}}\left|1,1;\frac{1}{2},-\frac{1}{2}\right\rangle$$ //

(4) (3)と同様に、

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle=\alpha\left|1,0;\frac{1}{2},\frac{1}{2}\right\rangle+\beta\left|1,1;\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$\left\langle\frac{1}{2},\frac{1}{2}\right|\left.\frac{1}{2},-\frac{1}{2}\right\rangle=\sqrt{\frac{1}{3}}\alpha-\sqrt{\frac{2}{3}}\beta=0$$

$$\begin{cases} \alpha^2+\beta^2=1 \\ \sqrt{\frac{1}{3}}\alpha-\sqrt{\frac{2}{3}}\beta=0 \end{cases} \rightarrow \quad \alpha=\sqrt{2}\beta$$

$$2\beta^2+\beta^2=1$$

$$\alpha=\sqrt{\frac{2}{3}} \qquad \beta=\sqrt{\frac{1}{3}}$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle=\sqrt{\frac{2}{3}}\left|1,0;\frac{1}{2},\frac{1}{2}\right\rangle+\sqrt{\frac{1}{3}}\left|1,1;\frac{1}{2},-\frac{1}{2}\right\rangle$$