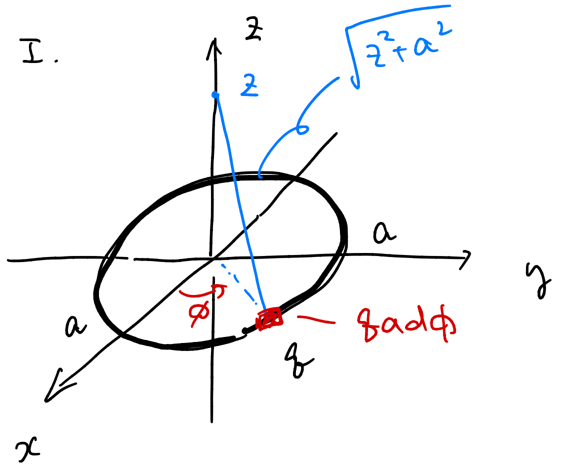
問題 2

I.



(1) 線密度 q なので

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{qa\,d\phi}{\sqrt{z^2+a^2}}$$

$$= \frac{1}{\cancel{4}\cancel{\pi}\varepsilon_0}\cdot\frac{2\cancel{\pi}qa}{\sqrt{z^2+a^2}}$$

$$= \frac{qa}{2\varepsilon_0\sqrt{z^2+a^2}} \quad //$$

(2)

$$\mathbf{E}(z) = -\mathrm{grad}\,\phi$$

$$= \frac{qa}{2\varepsilon_0}\frac{z}{(z^2+a^2)^{3/2}}\hat{\mathbf{e}}_z \quad //$$

$$\mathrm{grad} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

原点におかれた点電荷 Q' が受ける力は.

$$\mathbf{F} = Q'\mathbf{E}(z\to 0) = 0$$

(3)

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_C q\,ds'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\approx \frac{1}{4\pi\varepsilon_0}\int_{\cancel{C}\,0}^{2\pi} q\,\boxed{ds'}(\mathbf{r}-\mathbf{r}')\,a^{-3}\left[1+\frac{3}{a}(x\cos\varphi+y\sin\varphi)\right]$$

$ds' = a\,d\varphi$

$$E_x(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 a^2}\int_0^{2\pi}(x-a\cos\varphi)\left[1+\frac{3}{a}(x\cos\varphi+y\sin\varphi)\right]d\varphi$$

$$\int_0^{2\pi}(x-a\cos\varphi)\,d\varphi = \left[x\varphi - a\sin\varphi\right]_0^{2\pi} = 2\pi x$$

$$\int_0^{2\pi}(x-a\cos\varphi)(x\cos\varphi+y\sin\varphi)\,d\varphi$$

$$= \int_0^{2\pi}\left(x^2\cos\varphi + xy\sin\varphi - ax\underbrace{\cos^2\varphi}_{=\frac{\cos 2\varphi+1}{2}} + ay\sin\varphi\cos\varphi\right)d\varphi$$

$$= \int_0^{2\pi}\left(x^2\cos\varphi + xy\sin\varphi - \frac{ax}{2}\cos 2\varphi - \frac{ax}{2} + \frac{ay}{2}\sin 2\varphi\right)d\varphi$$

$$= \left[\cancel{x^2 \sin\varphi} - \cancel{xy\cos\varphi} - \cancel{\frac{ax}{4}\sin 2\varphi} - \frac{ax}{2}\varphi - \cancel{\frac{ay}{4}\cos 2\varphi} \right]_0^{2\pi}$$

$$= -ax\pi \qquad\qquad -3x\pi.$$

であるから、

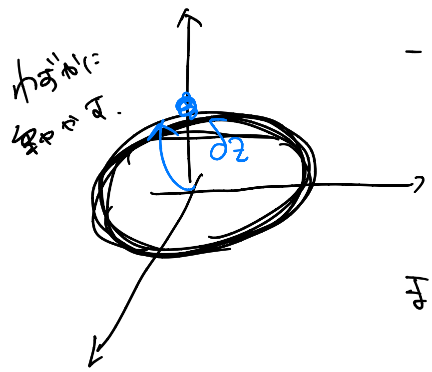
$$E_x = \frac{q}{4\pi\varepsilon_0 a^2}\left(\underbrace{2\pi x - 3x\pi}_{\to\, -\pi x}\right).$$

$$= -\frac{qx}{4\varepsilon_0 a^2}$$

xy の対称性より $E_y = -\dfrac{qy}{4\varepsilon_0 a^2}$

$$E_z = \frac{qa}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{z}{a^3}\, d\varphi$$

$$= \frac{qz}{2\varepsilon_0 a^2} \,//$$



$-q\mathbb{E}(\mathbb{r}=(0,0,\delta z))$ より、力 $\mathbb{F} = \left(0, 0, -\dfrac{q^2\delta z}{2\varepsilon_0 a^2}\right)$

なので点電荷が受ける力の向きは、

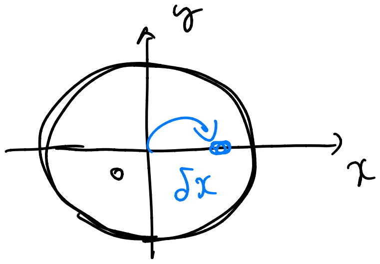
原点に戻る向きに力が働く.

また、xy平面上でずらすとき.

(対称性よりx方向のみずらしてみる)

$-q\mathbb{E}(\mathbb{r}=(\delta x,0,0))$ より 力$\mathbb{F} = \left(\dfrac{q^2\delta x}{4\varepsilon_0 a^2}, 0, 0\right)$

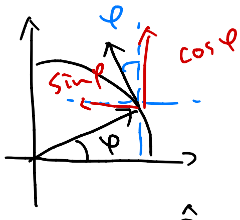
原点から離れる向きに力が働く.



II.

(4)
$$|\mathbb{r}-\mathbb{r}'|^{-1} = \left(|\mathbb{r}-\mathbb{r}'|^{2}\right)^{-\frac{1}{2}}$$
$$= \left(|\mathbb{r}|^2 - 2\mathbb{r}\cdot\mathbb{r}' + |\mathbb{r}'|^2\right)^{-\frac{1}{2}}$$
$$= |\mathbb{r}'|^{-1}\left\{\left(\frac{|\mathbb{r}|}{|\mathbb{r}'|}\right)^2 - \frac{2\mathbb{r}\cdot\mathbb{r}'}{|\mathbb{r}'|^2} + 1\right\}^{-\frac{1}{2}}$$
$$\approx |\mathbb{r}'|^{-1}\left(1 + \frac{\mathbb{r}\cdot\mathbb{r}'}{|\mathbb{r}'|}\right) = \frac{1}{|\mathbb{r}'|} + \frac{\mathbb{r}\cdot\mathbb{r}'}{|\mathbb{r}'|^3}$$

となるから.



$$\hat{\mathbb{e}}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$\mathbb{A} = \frac{\mu_0 I}{4\pi}\int_C \frac{d\mathbb{r}'}{|\mathbb{r}-\mathbb{r}'|}$$

$\mathbb{r}' = a\hat{\mathbb{e}}_r$

$d\mathbb{r}' = a\hat{\mathbb{e}}_\varphi d\varphi$

$$\approx \frac{\mu_0 I}{4\pi}\int_C d\mathbb{r}'\left\{\frac{1}{|\mathbb{r}'|} + \frac{\mathbb{r}\cdot\mathbb{r}'}{|\mathbb{r}'|^3}\right\}$$
$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} a\hat{\mathbb{e}}_\varphi d\varphi'\left\{\frac{1}{a} + \frac{a(x\cos\varphi' + y\sin\varphi')}{a^{\cancel{3}\,2}}\right\}$$

$$A_x = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} -\sin\varphi'\left(1 + \frac{x}{a}\cos\varphi' + \frac{y}{a}\sin\varphi'\right)d\varphi'$$
$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\left\{-\sin\varphi' - \frac{x}{2a}\sin 2\varphi' - \frac{y}{a}\left(\frac{1-\cos 2\varphi'}{2}\right)\right\}d\varphi'$$
$$= \frac{\mu_0 I}{4\pi}\left[-\frac{y}{4a}\varphi'\right]_0^{2\pi} = \frac{\mu_0 I}{4\cancel{\pi}}\cdot\left(-\frac{y}{\cancel{2}a}\cdot\cancel{2\pi}\right) = -\frac{\mu_0 I}{4a}y$$

$$A_y = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\cos\varphi'\left(1 + \frac{x}{a}\cos\varphi' + \frac{y}{a}\sin\varphi'\right)d\varphi'$$
$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\left\{\cos\varphi' + \frac{x}{a}\frac{1+\cos 2\varphi'}{2} + \frac{y}{2a}\sin 2\varphi'\right\}d\varphi'$$
$$= \frac{\mu_0 I}{4\pi}\left[\frac{x}{2a}\varphi'\right]_0^{2\pi} = \frac{\mu_0 I}{4\cancel{\pi}}\cdot\frac{x}{\cancel{2}a}\cdot\cancel{2\pi} = \frac{\mu_0 I}{4a}x$$

$$A_z = 0$$

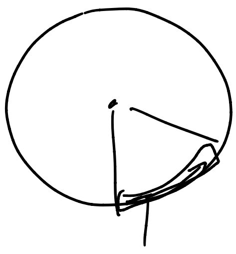
(5) $\mathbb{B} = \nabla \times \mathbb{A}$ より.

$$\mathbb{B} = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \\ e_x & e_y & e_z \end{vmatrix}$$

$$= \left(\cancel{\frac{\partial A_z}{\partial y}} - \cancel{\frac{\partial A_y}{\partial z}} \right) e_x + \left(\cancel{\frac{\partial A_x}{\partial z}} - \cancel{\frac{\partial A_z}{\partial x}} \right) e_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) e_z$$

$$= \left(\frac{\mu_0 I}{4a} + \frac{\mu_0 I}{4a} \right) e_z = \frac{\mu_0 I}{2a} e_z \quad //$$

(6)



$$\frac{dQ}{dt} = q a \dot{\varphi} = I \quad \cdots ①$$

$$I_c \dot{\varphi} = L \quad \cdots ②$$

①と②より $I = \dfrac{qaL}{I_c}$ //

(5)より $\mathbb{B} = \dfrac{\mu_0 q \cancel{a} L}{2 \cancel{a} I_c} \hat{e}_z$

$$= \frac{\mu_0 q L}{2 I_c} \hat{e}_z = \frac{\mu_0 q}{2 I_c} \mathbb{L} \quad (\because \mathbb{L} = L \hat{e}_z)$$

よって. $V = -\mathbb{M} \cdot \mathbb{B} = -\dfrac{\mu_0 q}{2 I_c} \mathbb{M} \cdot \mathbb{L}$ //

III.

(7) $\mathbb{B} = \nabla \times \mathbb{A}$

$$= \left(\frac{\partial \cancel{A_z}}{\partial \cancel{y}} - \frac{\partial A_y}{\partial z}\right) e_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial \cancel{A_z}}{\partial \cancel{x}}\right) e_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) e_x$$

ここで $\mathbb{A} = \frac{\mu_0 I a^2}{4r^3}(-y, x, 0)$ なので

$$= -\frac{\mu_0 I a^2}{4}\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) e_x - \frac{\mu_0 I a^2}{4}\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) e_y + \left\{\frac{\mu_0 I a^2}{4}\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\mu_0 I a^2}{4}\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)\right\} e_z$$

$$y\cdot\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\cdot\frac{\partial r}{\partial z} = y\cdot\left(-\frac{3}{r^4}\right)\cdot\frac{z}{r} = -\frac{3yz}{r^5}$$

$$x\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\cdot\frac{\partial r}{\partial z} = x\cdot\left(-\frac{3}{r^4}\right)\cdot\frac{z}{r} = -\frac{3xz}{r^5}$$

$$\frac{1}{r^3} + x\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = \frac{1}{r^3} + x\cdot\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\cdot\frac{\partial r}{\partial x} = \frac{1}{r^3} + x\cdot\left(-\frac{3}{r^4}\right)\cdot\frac{x}{r} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$

$$\frac{1}{r^3} - \frac{3y^2}{r^5}$$

$$= \frac{3\mu_0 I a^2 xz}{4r^5} e_x + \frac{3\mu_0 I a^2 yz}{4r^5} e_y + \frac{\mu_0 I a^2}{4}\left(\frac{2}{r^3} - \frac{3(x^2+y^2)}{r^5}\right) e_z$$

$$= \frac{2}{r^3} - \frac{3(r^2-z^2)}{r^5} = -\frac{1}{r^3} + \frac{3z^2}{r^5}$$

$$= \frac{3\mu_0 I a^2 xz}{4r^5} e_x + \frac{3\mu_0 I a^2 yz}{4r^5} e_y + \frac{\mu_0 I a^2}{4}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right) e_z \quad //$$

$\mathbb{m}_1 := (0, 0, \pi I a^2)$ とすると.

$$\mathbb{B}(\mathbb{r}) = \frac{\mu_0}{4\pi r^5}\left[3(\mathbb{m}_1\cdot\mathbb{r})\mathbb{r} - \mathbb{m}_1 r^2\right]$$

$$\left([\ ]\text{の中} = 3(\pi I a^2 z)(x e_x + y e_y + z e_z) - r^2 \pi I a^2 e_z = \pi I a^2 \left\{3xz e_x + 3yz e_y + (3z^2 - r^2) e_z\right\} \right)$$

$$= \frac{\mu_0}{4\cancel{\pi} r^5}\cdot\cancel{\pi} I a^2 \left\{3xz e_x + 3yz e_y + (3z^2 - r^2) e_z\right\}$$

$$= \frac{3\mu_0 I a^2 xz}{4r^5} \mathbb{e}_x + \frac{3\mu_0 I a^2 yz}{4r^5} \mathbb{e}_y + \frac{\mu_0 I a^2}{4}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right) \mathbb{e}_z$$ ■

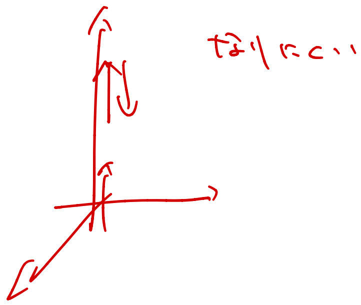
(8) $\mathbb{m}_2 := (0, 0, \pm \pi I a^2)$ より

$$V = -\mathbb{m}_2 \cdot \mathbb{B}(\mathbb{r} = (0,0,r))$$

$$= \mp \pi I a^2 \cdot \frac{\mu_0 I a^2}{4}\left(\frac{3r^2}{r^5} - \frac{1}{r^3}\right)$$

$$= \mp \pi I a^2 \frac{\mu_0 I a^2}{\cancel{4}\,2} \cdot \frac{\cancel{2}}{r^3}$$

$$= \mp \frac{\pi \mu_0 (I a^2)^2}{2r^3}$$



原点から距離 r のところにおいたとすると、

$$V = -\mathbb{m}_2 \cdot \mathbb{B}(\mathbb{r} = r)$$

$$= \mp \pi I a^2 \cdot \frac{\mu_0 I a^2}{4} \cdot \left(-\frac{1}{r^3}\right)$$

$$= \pm \frac{\pi \mu_0 (I a^2)^2}{4r^3}$$ //