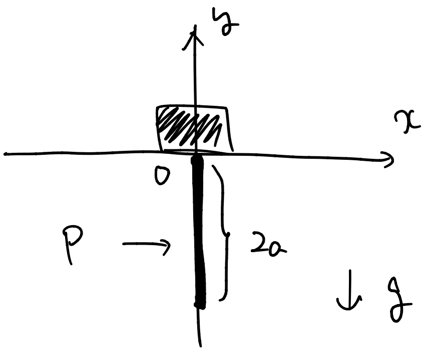
大阪大学 大学院

物理学専攻 2021

問題1

I.



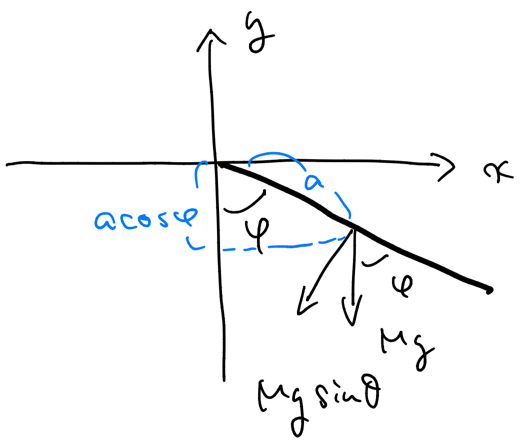
(1) O まわりの慣性モーメント

$$I_o = \frac{M}{2a}\int_0^{2a} y^2\,dy$$

$$= \frac{M}{2a}\left[\frac{1}{3}y^3\right]_0^{2a}$$

$$= \frac{M}{\cancel{2a}}\cdot\frac{1}{3}\cdot(2a)^{\cancel{3}\,2} = \frac{4}{3}Ma^2$$ //

(2) 回転運動の運動方程式は.



$$I_o\ddot{\varphi} = -a\cdot Mg\sin\varphi$$

$$\sim -aMg\varphi$$

$$\ddot{\varphi} = -\frac{Mga}{I_o}\varphi$$

ゆえに角振動数 ω は. $\omega = \sqrt{\dfrac{Mga}{I_o}}$ //

(3) $\mathbb{L} = \mathbb{r}\times\mathbb{P}$ より $\underbrace{I\dot{\varphi} = aP}_{z成分に注目すると}$ がいえる。 $\dot{\varphi} = \dfrac{aP}{I}$

力学的エネルギー保存則より

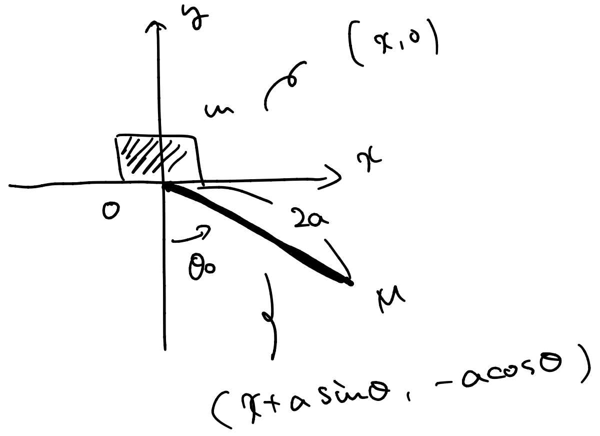
$$\frac{1}{2}I\dot{\varphi}^2 = Mga(1-\cos\varphi_0) \sim Mga\left(1-1+\frac{1}{2}\varphi_0^2\right)$$

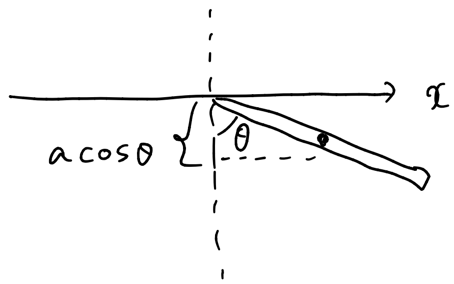
$$\cancel{\frac{1}{2}}I\left(\frac{aP}{I}\right)^2 = Mga\cancel{\frac{1}{2}}\varphi_0^2$$

$$\varphi_0 = \sqrt{\left(\frac{aP}{I}\right)^2 I\Big/Mga} = P\sqrt{\frac{a}{MgI}}$$ //

II.

(4)





棒Aの速度成分は $\dot{x}' = \dot{x} + a\dot{\theta}\cos\theta$, $\dot{y}' = a\dot{\theta}\sin\theta$ より。

この系の運動エネルギーTは、

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\{(\dot{x}+a\dot{\theta}\cos\theta)^2 + (a\dot{\theta}\sin\theta)^2\} + \frac{1}{2}I\dot{\theta}^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{x}^2 + M\dot{x}a\dot{\theta}\cos\theta + (a\dot{\theta})^2 + \frac{1}{2}I\dot{\theta}^2$$

$$U = \underbrace{0}_{\text{台Aのポテンシャル}} + \underbrace{Mg(-a\cos\theta)}_{\text{棒のポテンシャル}} = -Mga\cos\theta$$

(5) $L = T - U$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{x}^2 + M\dot{x}a\dot{\theta}\cos\theta + \frac{1}{2}M(a\dot{\theta})^2 + \frac{1}{2}I\dot{\theta}^2 + Mga\cos\theta$$

$$= \underline{\frac{1}{2}(m+M)\dot{x}^2} + M\dot{x}a\dot{\theta}\left(\underline{1} - \cancel{\frac{1}{2}\theta^2}\right) + \frac{1}{2}M\underline{(a\dot{\theta})^2} + \underline{\frac{1}{2}I\dot{\theta}^2}$$
$$+ Mga\left(1 - \underline{\frac{1}{2}\theta^2}\right)$$

3次以上より。

$$= \frac{1}{2}(m+M)\dot{x}^2 + \left(\frac{1}{2}Ma^2 + \frac{1}{2}I\right)\dot{\theta}^2 + Ma\dot{x}\dot{\theta}$$
$$+ Mga - \frac{1}{2}Mga\theta^2.$$

定数hとして $L' = L + h$ は成立するので

$$L = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}(Ma^2+I)\dot{\theta}^2 + Ma\dot{x}\dot{\theta} - \frac{1}{2}Mga\theta^2$$

//

(6) x, θ に関するオイラー・ラグランジュ方程式は

$$\begin{cases} \cdot \ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) - \dfrac{\partial L}{\partial x} = 0 \\ \cdot \ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = 0 \end{cases}$$

であるから、

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = (m+M)\ddot{x} + Ma\ddot{\theta} \\ \dfrac{\partial L}{\partial x} = 0 \end{cases}$$

$$\longrightarrow (m+M)\ddot{x} + Ma\ddot{\theta} = 0 \quad \text{———} ☆$$

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) = (Ma^2+I)\ddot{\theta} + Ma\ddot{x} \\ \dfrac{\partial L}{\partial \theta} = -Mga\theta \end{cases}$$

$$\longrightarrow (Ma^2+I)\ddot{\theta} + Ma\ddot{x} = -Mga\theta \quad \text{———} ☆☆$$

(7) $m \to \infty$ のとき ☆ は、$m\ddot{x} = 0$ となり、台Aは、動かない。つまり $\ddot{x} = 0$ より、☆☆ は、

$$(Ma^2+I)\ddot{\theta} = -Mga\theta$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{Mga}{Ma^2+I}\theta$$

ゆえに、角振動数 $\sqrt{\dfrac{Mga}{Ma^2+I}}$ の振動をする。

(8) $m = 0$ のとき、☆ は、$M\ddot{x} + Ma\ddot{\theta} = 0$ 、$\Rightarrow \ \ddot{x} = -a\ddot{\theta}$

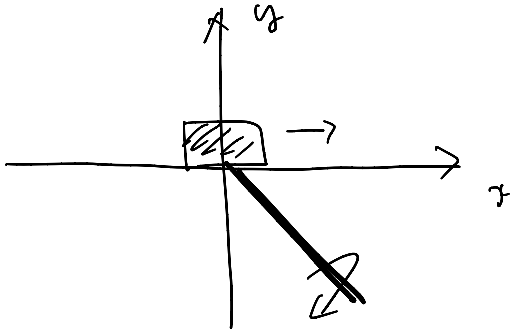
☆☆ に代入すると、

$$(Ma^2+I)\ddot{\theta} + Ma(-a\ddot{\theta}) = -Mga\theta$$

$$I\ddot{\theta} = -Mga\theta$$

$$\ddot{\theta} = -\frac{Mga}{I}\theta$$

角振動数 $\sqrt{\frac{Mga}{I}}$ の振動をする。



この系の外力はないので
この系の重心は
静止している。

台Aと棒は
逆向きの動きをする。

(9) 微分方程式：

$$\begin{cases} (m+M)\ddot{x} + Ma\ddot{\theta} = 0 & \text{—— ☆} \\ (Ma^2+I)\ddot{\theta} + Ma\ddot{x} = -Mga\theta & \text{—— ☆☆} \end{cases}$$

☆を☆☆に代入すると、

$$(Ma^2+I)\ddot{\theta} + Ma\left(-\frac{Ma}{m+M}\ddot{\theta}\right) = -Mga\theta$$

$$\frac{Mma^2}{m+M}\ddot{\theta} = -Mga\theta$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{mga}{m+M}\theta \text{ で } \omega := \sqrt{\frac{mga}{m+M}}$$

$$\therefore \quad \theta(t) = Ae^{-i\omega t} + Be^{i\omega t}$$ （A、Bは任意定数）

$$\theta(0) = A + B = \theta_0$$

$$\dot{\theta}(t) = i\omega(B-A) = 0$$ より $A=B$.

$$\theta(t) = \theta_0 \frac{e^{-i\omega t} + e^{i\omega t}}{2}$$

$$= \theta_0 \cos(\omega t)$$ //

また、 $\ddot{\theta}(t) = -\theta_0 \omega^2 \cos(\omega t)$ で ㊍ に代入すると、

$$(m+M)\ddot{x} = -\theta_0 \omega^2 M a \cos(\omega t)$$

$$\ddot{x}(t) = -\frac{M}{m+M} a \theta_0 \omega^2 \cos(\omega t)$$

$$\dot{x}(t) = -\frac{M}{m+M} a \theta_0 \omega \sin(\omega t) + C$$

$$x(t) = \frac{M}{m+M} a \theta_0 \cos(\omega t) + Ct + D$$

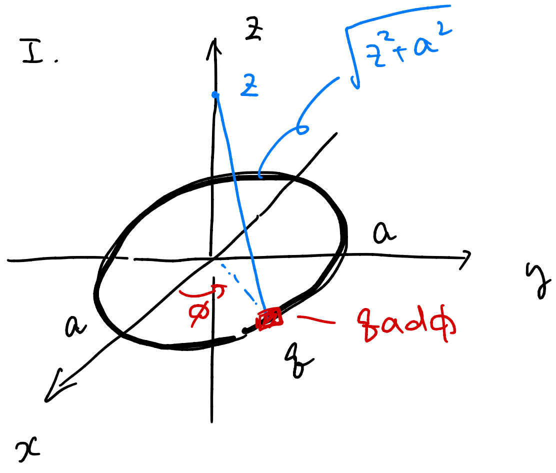
$$x(0) = \frac{M}{m+M} a \theta_0 + D = 0 \qquad \therefore\ D = -\frac{M}{m+M} a \theta_0$$

$$\dot{x}(0) = C = 0$$

以上より $x(t) = \dfrac{M}{m+M} a \theta_0 \cos(\omega t) - \dfrac{M}{m+M} a \theta_0$ //

問題 2

I.



(1) 線密度 q なので

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{q a\, d\phi}{\sqrt{z^2+a^2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{2\pi q a}{\sqrt{z^2+a^2}}$$

$$= \frac{q a}{2\varepsilon_0\sqrt{z^2+a^2}}$$ //

(2) $$E(z) = -\,\mathrm{grad}\,\phi$$

$$\mathrm{grad} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$= \frac{q a}{2\varepsilon_0}\frac{z}{(z^2+a^2)^{3/2}}\hat{e}_z$$ //

原点におかれた点電荷 Q' が受ける力は.

$$F = Q' E(z\to 0) = 0$$

(3) $$E(r) = \frac{1}{4\pi\varepsilon_0}\int_C q\, ds' \frac{r - r'}{|r - r'|^3}$$

$$\approx \frac{1}{4\pi\varepsilon_0}\int_{\cancel{C}\,0}^{2\pi} q\, \underbrace{\boxed{ds'}}_{a\,d\varphi}(r - r')\, a^{-3}\left[1 + \frac{3}{a}(x\cos\varphi + y\sin\varphi)\right]$$

$$E_x(r) = \frac{q}{4\pi\varepsilon_0 a^2}\int_0^{2\pi}(x - a\cos\varphi)\left[1 + \frac{3}{a}(x\cos\varphi + y\sin\varphi)\right]d\varphi$$

$$\int_0^{2\pi}(x - a\cos\varphi)\,d\varphi = \left[x\varphi - a\sin\varphi\right]_0^{2\pi} = 2\pi x$$

$$\int_0^{2\pi}(x - a\cos\varphi)(x\cos\varphi + y\sin\varphi)\,d\varphi$$

$$= \int_0^{2\pi}\left(x^2\cos\varphi + xy\sin\varphi - a x\underbrace{\cos^2\varphi}_{=\frac{\cos 2\varphi + 1}{2}} + a y\sin\varphi\cos\varphi\right)d\varphi$$

$$= \int_0^{2\pi}\left(x^2\cos\varphi + xy\sin\varphi - \frac{ax}{2}\cos 2\varphi - \frac{ax}{2} + \frac{ay}{2}\sin 2\varphi\right)d\varphi$$

$$= \left[\cancel{x^2 \sin\varphi} - \cancel{xy\cos\varphi} - \cancel{\frac{ax}{4}\sin 2\varphi} - \frac{ax}{2}\varphi - \cancel{\frac{ay}{4}\cos 2\varphi} \right]_0^{2\pi}$$

$$= -ax\pi \qquad\qquad -3x\pi.$$

であるから.

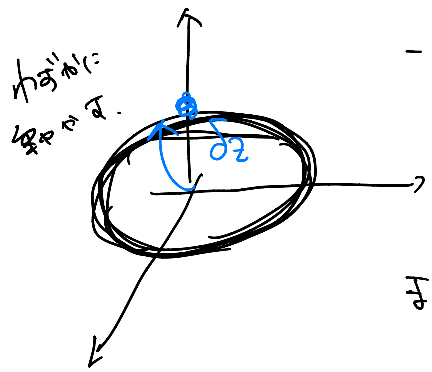
$$E_x = \frac{q}{4\pi\varepsilon_0 a^2}\left(\underbrace{2\pi x - 3x\pi}_{\to\, -\pi x}\right).$$

$$= -\frac{qx}{4\varepsilon_0 a^2}$$

xyの対称性より $E_y = -\dfrac{qy}{4\varepsilon_0 a^2}$

$$E_z = \frac{qa}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{z}{a^3}\, d\varphi$$

$$= \frac{qz}{2\varepsilon_0 a^2}$$



$-qE(r=(0,0,\delta z))$ より、力 $F = \left(0, 0, -\dfrac{q^2\delta z}{2\varepsilon_0 a^2}\right)$

なので点電荷が受ける力の向きは、

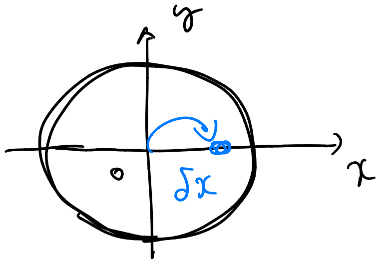
原点に戻る向きに力が働く.

また、xy平面上でずらすとき.

(対称性よりx方向のみずらしてみる)

$-qE(r=(\delta x,0,0))$ より 力 $F = \left(\dfrac{q^2\delta x}{4\varepsilon_0 a^2}, 0, 0\right)$

原点から離れる向きに力が働く.



II.

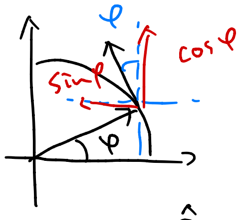
(4) $$|\mathbb{r}-\mathbb{r}'|^{-1} = |\mathbb{r}-\mathbb{r}'|^{2\cdot\left(-\frac{1}{2}\right)}$$

$$= \left(|\mathbb{r}|^2 - 2\mathbb{r}\cdot\mathbb{r}' + |\mathbb{r}'|^2\right)^{-\frac{1}{2}}$$

$$= |\mathbb{r}'|^{-1}\left\{\left(\frac{|\mathbb{r}|}{|\mathbb{r}'|}\right)^2 - \frac{2\mathbb{r}\cdot\mathbb{r}'}{|\mathbb{r}'|^2} + 1\right)\right\}^{-\frac{1}{2}}$$

$$\approx |\mathbb{r}'|^{-1}\left(1 + \frac{\mathbb{r}\cdot\mathbb{r}'}{|\mathbb{r}'|}\right) = \frac{1}{|\mathbb{r}'|} + \frac{\mathbb{r}\cdot\mathbb{r}'}{|\mathbb{r}'|^3}$$

となるから.



$$\mathbb{A} = \frac{\mu_0 I}{4\pi}\int_C \frac{d\mathbb{r}'}{|\mathbb{r}-\mathbb{r}'|}$$

$\hat{\mathbb{e}}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$

$\mathbb{r}' = a\hat{\mathbb{e}}_r$

$d\mathbb{r}' = a\hat{\mathbb{e}}_\varphi d\varphi$

$$\approx \frac{\mu_0 I}{4\pi}\int_C d\mathbb{r}'\left\{\frac{1}{|\mathbb{r}'|} + \frac{\mathbb{r}\cdot\mathbb{r}'}{|\mathbb{r}'|^3}\right\}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} a\hat{\mathbb{e}}_\varphi d\varphi'\left\{\frac{1}{a} + \frac{a(x\cos\varphi' + y\sin\varphi')}{a^{\cancel{3}2}}\right\}$$

$$A_x = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} -\sin\varphi'\left(1 + \frac{x}{a}\cos\varphi' + \frac{y}{a}\sin\varphi'\right)d\varphi'$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\left\{-\sin\varphi' - \frac{x}{2a}\sin 2\varphi' - \frac{y}{a}\left(\frac{1-\cos 2\varphi'}{2}\right)\right\}d\varphi'$$

$$= \frac{\mu_0 I}{4\pi}\left[-\frac{y}{4a}\varphi'\right]_0^{2\pi} = \frac{\mu_0 I}{4\cancel{\pi}}\cdot\left(-\frac{y}{\cancel{2}a}\cdot\cancel{2\pi}\right) = -\frac{\mu_0 I}{4a}y$$

$$A_y = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\cos\varphi'\left(1 + \frac{x}{a}\cos\varphi' + \frac{y}{a}\sin\varphi'\right)d\varphi'$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\left\{\cos\varphi' + \frac{x}{a}\frac{1+\cos 2\varphi'}{2} + \frac{y}{2a}\sin 2\varphi'\right\}d\varphi'$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{x}{2a}\varphi'\right]_0^{2\pi} = \frac{\mu_0 I}{4\cancel{\pi}}\cdot\frac{x}{\cancel{2}a}\cdot\cancel{2\pi} = \frac{\mu_0 I}{4a}x$$

$$A_z = 0$$

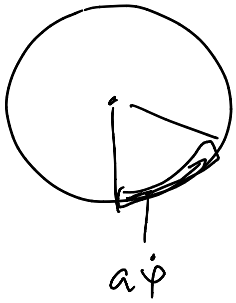
(5) $\mathbb{B} = \nabla \times \mathbb{A}$ より.

$$\mathbb{B} = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \\ e_x & e_y & e_z \end{vmatrix}$$

$$= \left(\cancel{\frac{\partial A_z}{\partial y}} - \cancel{\frac{\partial A_y}{\partial z}} \right) e_x + \left(\cancel{\frac{\partial A_x}{\partial z}} - \cancel{\frac{\partial A_z}{\partial x}} \right) e_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) e_z$$

$$= \left(\frac{\mu_0 I}{4a} + \frac{\mu_0 I}{4a} \right) e_z = \frac{\mu_0 I}{2a} e_z \quad //$$

(6)



$$\frac{dQ}{dt} = q a \dot{\varphi} = I \quad \cdots ①$$

$$I_c \dot{\varphi} = L \quad \cdots ②$$

① と ② より $I = \dfrac{qaL}{I_c}$ //

(5) より
$$\mathbb{B} = \frac{\mu_0 q \cancel{a} L}{2 \cancel{a} I_c} \hat{e}_z$$

$$= \frac{\mu_0 q L}{2 I_c} \hat{e}_z = \frac{\mu_0 q}{2 I_c} \mathbb{L} \quad (\because \mathbb{L} = L \hat{e}_z)$$

よって. $V = -\mathbb{M} \cdot \mathbb{B} = -\dfrac{\mu_0 q}{2 I_c} \mathbb{M} \cdot \mathbb{L}$ //

III.

(7) $\mathbb{B} = \nabla \times \mathbb{A}$

$$= \left(\cancel{\frac{\partial A_z}{\partial y}} - \frac{\partial A_y}{\partial z}\right) e_x + \left(\frac{\partial A_x}{\partial z} - \cancel{\frac{\partial A_z}{\partial x}}\right) e_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) e_x$$

ここで $\mathbb{A} = \dfrac{\mu_0 I a^2}{4 r^3}(-y, x, 0)$ なので

$$y \cdot \frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\cdot \frac{\partial r}{\partial z} = y \cdot \left(-\frac{3}{r^4}\right)\cdot \frac{z}{r} = -\frac{3yz}{r^5}$$

$$= -\frac{\mu_0 I a^2}{4}\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) e_x - \frac{\mu_0 I a^2}{4}\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) e_y + \left\{\frac{\mu_0 I a^2}{4}\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\mu_0 I a^2}{4}\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)\right\} e_z$$

$$x \frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\cdot\frac{\partial r}{\partial z} = x\cdot\left(-\frac{3}{r^4}\right)\cdot\frac{z}{r} = -\frac{3xz}{r^5}$$

$$\frac{1}{r^3} + x\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = \frac{1}{r^3} + x\cdot\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\cdot\frac{\partial r}{\partial x} = \frac{1}{r^3} + x\cdot\left(-\frac{3}{r^4}\right)\cdot\frac{x}{r} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$

$$\frac{1}{r^3} - \frac{3y^2}{r^5}$$

$$= \frac{3\mu_0 I a^2 xz}{4r^5} e_x + \frac{3\mu_0 I a^2 yz}{4r^5} e_y + \frac{\mu_0 I a^2}{4}\left(\frac{2}{r^3} - \frac{3(x^2+y^2)}{r^5}\right) e_z$$

$$= \frac{2}{r^3} - \frac{3(r^2 - z^2)}{r^5} = -\frac{1}{r^3} + \frac{3z^2}{r^5}$$

$$= \frac{3\mu_0 I a^2 xz}{4r^5} e_x + \frac{3\mu_0 I a^2 yz}{4r^5} e_y + \frac{\mu_0 I a^2}{4}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right) e_z$$

//

$\mathbb{m}_1 := (0, 0, \pi I a^2)$ とすると.

$$\mathbb{B}(\mathbb{r}) = \frac{\mu_0}{4\pi r^5}\left[3(\mathbb{m}_1\cdot\mathbb{r})\mathbb{r} - \mathbb{m}_1 r^2\right]$$

([ ] の中 $= 3(\pi I a^2 z)(x e_x + y e_y + z e_z) - r^2 \pi I a^2 e_z$

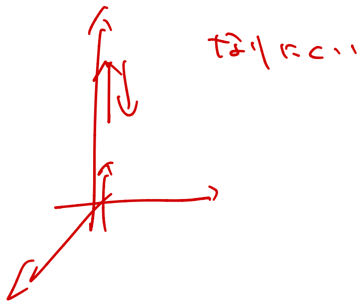
$= \pi I a^2 \{3xz e_x + 3yz e_y + (3z^2 - r^2) e_z\}$)

$$= \frac{\mu_0}{4\cancel{\pi} r^5}\cdot \cancel{\pi} I a^2 \{3xz e_x + 3yz e_y + (3z^2 - r^2) e_z\}$$

$$
= \frac{3\mu_0 I a^2 xz}{4r^5}\mathbf{e}_x + \frac{3\mu_0 I a^2 yz}{4r^5}\mathbf{e}_y + \frac{\mu_0 I a^2}{4}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\mathbf{e}_z \qquad \blacksquare
$$

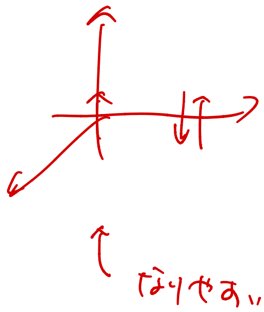
(8) $\mathbf{m}_2 := (0, 0, \pm\pi I a^2)$ より

$$
\begin{aligned}
V &= -\mathbf{m}_2 \cdot \mathbf{B}(\mathbf{r} = (0,0,r)) \\
&= \mp \pi I a^2 \cdot \frac{\mu_0 I a^2}{4}\left(\frac{3r^2}{r^5} - \frac{1}{r^3}\right) \\
&= \mp \pi I a^2 \frac{\mu_0 I a^2}{\cancel{4}_{2}} \cdot \frac{\cancel{2}}{r^3} \\
&= \mp \frac{\pi \mu_0 (I a^2)^2}{2r^3}
\end{aligned}
$$



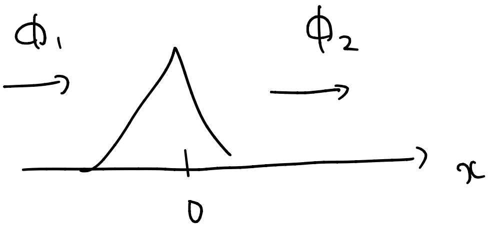
原点から距離 r のところにおいたとすると、

$$
\begin{aligned}
V &= -\mathbf{m}_2 \cdot \mathbf{B}(r = r) \\
&= \mp \pi I a^2 \cdot \frac{\mu_0 I a^2}{4} \cdot \left(-\frac{1}{r^3}\right) \\
&= \pm \frac{\pi \mu_0 (I a^2)^2}{4r^3} \quad //
\end{aligned}
$$



問題 3

I.



(1) $x \to 0$ で一致することから.

$$\Phi_1(0) = \Phi_2(0) \quad \cdots\cdots ①$$

がいえる。

$$\therefore \quad 1 + r = t$$

(2) Schrödinger方程式の両辺を $-\varepsilon \sim \varepsilon$ で積分すると.

$$\int_{-\varepsilon}^{\varepsilon} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Phi(x)\,dx + \int_{-\varepsilon}^{\varepsilon} V(x)\Phi(x)\,dx = \int_{-\varepsilon}^{\varepsilon} E\phi(x)\,dx.$$

$\varepsilon \to 0$ で 0.

$$-\frac{\hbar^2}{2m}\left[\frac{d}{dx}\Phi(x)\right]_{-\varepsilon}^{\varepsilon}$$

$$= -\frac{\hbar^2}{2m}\left(\frac{d}{dx}\Phi(x)\Big|_{x=\varepsilon} - \frac{d}{dx}\Phi(x)\Big|_{x=-\varepsilon}\right)$$

$$\int_{-\varepsilon}^{\varepsilon} -V_0\,\delta(x)\,\Phi(x)\,dx$$

$$= -V_0\Phi(0)$$

以上より

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dx}\Phi(x)\Big|_{x=\varepsilon} - \frac{d}{dx}\Phi(x)\Big|_{x=-\varepsilon}\right) = V_0\Phi(0)$$

$$\frac{d}{dx}\Phi(x)\Big|_{x=\varepsilon} = \underline{\underline{iq}}\,t\,e^{iq\varepsilon}$$

$$\frac{d}{dx}\Phi(x)\Big|_{x=-\varepsilon} = \underline{\underline{iq}}\,e^{-iq\varepsilon} - \underline{\underline{iq}}\,r\,e^{+iq\varepsilon}$$

$\varepsilon \to 0$ において.

$$-\frac{\hbar^2}{2m}iq\left(t - (1-r)\right) = V_0 t$$

$$\Rightarrow \quad t + r = -\frac{2mV_0}{\hbar^2 i q}t + 1 = \frac{2i}{\omega}t + 1$$

$= \frac{1}{\omega}$

//

(3)

$$\left.\begin{aligned} t+r &= \frac{2i}{\omega}t+1 \\ 1+r &= t \end{aligned}\right\}$$ より

$$t+t-1 = \frac{2i}{\omega}t+1$$

$$\left(\frac{\cancel{2}\omega-\cancel{2}i}{\omega}\right)t = \cancel{2} \qquad t = \frac{\omega}{\omega-i}$$

$$|t|^2 = \frac{\omega^2}{(\omega-i)(\omega+i)} = \frac{\omega^2}{\omega^2+1}$$

また $r = t-1 = \dfrac{\omega}{\omega-i} - \dfrac{\omega-i}{\omega-i} = \dfrac{i}{\omega-i}$

$$|r|^2 = \frac{1}{(\omega-i)(\omega+i)} = \frac{1}{\omega^2+1}$$

//

(4) $E\to 0$ のとき. $q = \dfrac{\sqrt{2mE}}{\hbar} \to 0$

$$|t|^2 \to 0 \quad , \quad |r|^2 \to 1$$

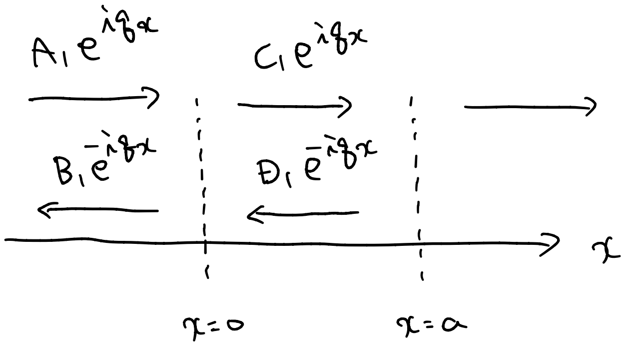
$E\to\infty$ のとき $q\to\infty$

$$|t|^2 = \frac{1}{1+\frac{1}{\omega^2}} \to 1 \quad , \quad |r|^2 \to 0$$

次に $V_0 \to -V_0$ となったとき. $\omega^2 \to \omega^2$ なので 不変。

II.

(5)



上図より B_1は A_1の反射分と D_1の透過分の和で表せる。C_1は A_1の透過分と D_1の反射分の和で表せる。ゆえに、

$$\begin{cases} B_1 = rA_1 + tD_1 & \cdots\cdots ② \\ C_1 = tA_1 + rD_1 & \cdots ③ \end{cases}$$

(6)

$\psi_1(0) = \psi_2(0)$ より $A_1 + B_1 = C_1 + D_1 \quad \cdots ④$

ゆえに、③と④より

$$C_1 = tA_1 + r(A_1 + B_1 - C_1)$$

$$(1+r)C_1 = (t+r)A_1 + rB_1$$

$$\therefore\ C_1 = \frac{t+r}{\underbrace{1+r}_{=t}} A_1 + \frac{r}{\underbrace{1+r}_{=t}} B_1$$

また、②より

$$D_1 = -\frac{r}{t} A_1 + \frac{1}{t} B_1$$

以上より、

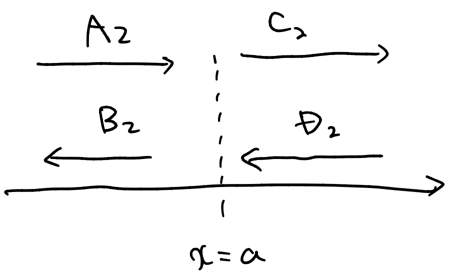
$$\begin{pmatrix} C_1 \\ D_1 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{t+r}{t} & \frac{r}{t} \\ -\frac{r}{t} & \frac{1}{t} \end{pmatrix}}_{=Z} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$$

(7)

$$\psi_2(x) = \underbrace{A_2 e^{-iqa}}_{=C_1} e^{iqx} + \underbrace{B_2 e^{iqa}}_{=D_1} e^{-iqx}$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \underbrace{\begin{pmatrix} e^{iqa} & 0 \\ 0 & e^{-iqa} \end{pmatrix}}_{=Y} \begin{pmatrix} C_1 \\ D_1 \end{pmatrix} \quad //$$

(8)



$$\begin{pmatrix} C_2 \\ D_2 \end{pmatrix} = Z \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = ZY \begin{pmatrix} C_1 \\ D_1 \end{pmatrix} = ZYZ \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$$

上図より.

$$C_2 = t A_2 + r D_2$$

$$B_2 = r A_2 + t D_2$$

同様にして,

$$\begin{pmatrix} C_2 \\ D_2 \end{pmatrix} = \underbrace{ZYZ}_{=X} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$$

(9)

$$\begin{pmatrix} t_2 \\ 0 \end{pmatrix} = ZYZ \begin{pmatrix} 1 \\ r_2 \end{pmatrix}$$

$$\frac{t+r}{t} = \frac{1}{\frac{\omega}{\omega-i}} = \frac{\omega-i}{\omega}, \quad \frac{r}{t} = \frac{\frac{i}{\omega-i}}{\frac{\omega}{\omega-i}} = \frac{i}{\omega}$$

$$-\frac{r}{t} = -\frac{\frac{i}{\omega-i}}{\frac{\omega}{\omega-i}} = -\frac{i}{\omega}, \quad \frac{1}{t} = \frac{1}{\frac{\omega}{\omega-i}} = \frac{\omega-i}{\omega}$$

$$Z = \begin{pmatrix} \frac{t+r}{t} & \frac{r}{t} \\ -\frac{r}{t} & \frac{1}{t} \end{pmatrix} = \begin{pmatrix} \frac{\omega-i}{\omega} & \frac{i}{\omega} \\ -\frac{i}{\omega} & \frac{\omega-i}{\omega} \end{pmatrix}$$

$$Y = \begin{pmatrix} e^{i\cancel{g}\cdot 2n\pi/\cancel{g}} & 0 \\ 0 & e^{-ig\cdot 2n\pi/g} \end{pmatrix} = I \quad (\text{単位行列})$$

$$Z^2 = \begin{pmatrix} \frac{\omega+2i}{\omega} & \frac{2i}{\omega} \\ -\frac{2i}{\omega} & \frac{\omega-2i}{\omega} \end{pmatrix}$$

$$\begin{pmatrix} t_2 \\ 0 \end{pmatrix} = \underbrace{Z Y Z}_{= Z^2} \begin{pmatrix} 1 \\ r_2 \end{pmatrix}$$ より

$$t_2 = \frac{\omega+2i}{\omega} + \frac{2i}{\omega} r_2 \quad \cdots ⑤$$

$$0 = -\frac{2i}{\omega} + \frac{\omega-2i}{\omega} r_2 \quad \cdots ⑥$$

⑥ より

$$\frac{\omega-2i}{\omega} r_2 = \frac{2i}{\omega}$$

$$r_2 = \frac{2i}{\omega-2i}$$

⑤ より

-4

$$t_2 = \frac{\omega+2i}{\omega} + \frac{2i}{\omega} \cdot \frac{2i}{\omega-2i}$$

$$= \frac{\omega^2+4-4}{\omega(\omega-2i)}$$

$$= \frac{\omega}{\omega-2i}$$

$$|r_2|^2 = \frac{2i}{\omega-2i} \cdot \frac{-2i}{\omega+2i} = \frac{4}{\omega^2+4}$$

$$|t_2|^2 = \frac{\omega}{\omega-2i} \cdot \frac{\omega}{\omega+2i} = \frac{\omega^2}{\omega^2+4}$$

- $x=0$ にくらべて $x=a$ では反射率が高く透過率が小さい。

- ポテンシャルの壁を2つおくことはポテンシャルの高さを2倍にするのと等しい
  (ω において $V_0 \to 2V_0$ と同じ)

$\boxed{問題4}$

(1) $dG = dF + VdP + PdV$

$= -SdT - \cancel{PdV} + \mu dN + VdP + \cancel{PdV}$

$= -SdT + VdP + \mu dN$

(2) 適当な a を用いて

$$aG(T, p, N) = G(T, p, aN)$$

が成り立ち、両辺を微分すると

$$G(T, p, N) = \frac{\partial (aN)}{\partial a} \cdot \frac{\partial G}{\partial (aN)}$$

$$= N\mu$$

$\therefore \quad G = \mu N$

また $dG = \underbrace{\left(\frac{\partial G}{\partial T}\right)}_{=-S} dT + \underbrace{\left(\frac{\partial G}{\partial P}\right)}_{=V} dP + \underbrace{\left(\frac{\partial G}{\partial N}\right)}_{=\mu} dN$

$\mu = \frac{G}{N}$ より $d\mu = \frac{1}{N}\underbrace{\left(\frac{\partial G}{\partial T}\right)}_{=-S} dT + \frac{1}{N}\underbrace{\left(\frac{\partial G}{\partial P}\right)}_{=V} dP$

なので $d\mu = -\frac{S}{N} dT + \frac{V}{N} dP$

$= -s\,dT + v\,dp$

//

Ⅱ.

(3)

$$Z(T,V,N) = \frac{V^N}{h^{3N} N!} \int d^3p_1 d^3p_2 \cdots d^3p_N \exp\left(-\frac{1}{k_BT}\sum_{i=1}^{N}\frac{p_i^2}{2m}\right)$$

$$(\text{積分部分}) = \int d^3p_1 \cdots d^3p_N \exp\left(-\frac{1}{2mk_BT}\sum_{i=1}^{N} p_i^2\right)$$

$$= \Bigg\{ \underbrace{\int d^3p_k \exp\left(-\frac{p_k^2}{2mk_BT}\right)}_{=(2mk_BT\pi)^{\frac{3}{2}}} \Bigg\}^N$$

$$= (2mk_BT\pi)^{\frac{3}{2}N}$$

$$Z(T,V,N) = \frac{V^N}{h^{3N}N!}(2mk_BT\pi)^{\frac{3}{2}N}$$

$$= \frac{V^N}{N!}\left(\frac{\sqrt{2\pi mk_BT}}{h}\right)^{3N}$$

$$= \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N$$

ヘルムホルツの自由エネルギーFは、

$$F(T,V,N) = -k_BT\log Z$$

$$= -k_BT\underbrace{\log\frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N}_{*}$$

$$* = N\log\left(\frac{V}{\lambda^3}\right) - \log N!$$

$$= N\log\left(\frac{V}{\lambda^3}\right) - N\log N + N$$

$$\therefore \quad F = k_BTN\left(\log\frac{\lambda^3}{V} + \log N - 1\right) \quad //$$

(4)

$$G = F + PV$$

$$= k_BTN\left(\log\frac{\lambda^3}{V} + \log N - 1\right) + PV$$

また、$\mu = \frac{G}{N}$

$$= k_BT\left(\log\frac{\lambda^3}{V} + \log N - 1\right) + \frac{PV}{N}$$

理想気体なので
$PV = nRT$
$= \frac{N}{N_A} RT$
$= \frac{N}{N_A} N_A k_B T$
$v = \frac{k_B T}{P}$

$$= k_B T \left(\log \lambda^3 - \log V + \log N - 1 \right) + P \cdot \frac{V}{N}$$

$$= k_B T \left(3 \log \lambda - \log \boxed{\frac{V}{N}} - 1 \right) + P \cdot \boxed{\frac{V}{N}}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \; v \qquad\qquad\qquad\quad \| \; v$

練習.

ここで $d\mu(T,P) = \left(\frac{\partial \mu}{\partial T}\right)_P dT + \underbrace{\left(\frac{\partial \mu}{\partial P}\right)_T}_{= v} dP$ より $\frac{\partial \mu}{\partial P}$ を計算する。

$$\left(\frac{\partial \mu}{\partial P}\right)_T = k_B T \left(-\frac{1}{v} \left(\frac{\partial v}{\partial P}\right)_T \right) + \cancel{v} + P \left(\frac{\partial v}{\partial P}\right)_T = \cancel{v}$$

$\therefore \quad -\frac{k_B T}{v} + P = 0$ より $v = \frac{k_B T}{P}$ //

ゆえに. $\mu = k_B T \left(3 \log \lambda - \log \frac{k_B T}{P} - 1 \right) + k_B T$

$$= 3 k_B T \log \left(\frac{\lambda P}{k_B T} \right)$$ //

(5) ボーズ粒子の$\underbrace{\text{状態密度}}_{}$は.
単位体積あたりの

$$D(\varepsilon) = \frac{1}{V} \sum_{k} \delta(\varepsilon - \varepsilon_k)$$

$$= \frac{1}{V} \int \frac{d^3k}{\left(\frac{2\pi}{L}\right)^3} \delta(\varepsilon - \varepsilon_k)$$

$$= \frac{1}{\cancel{V}} \cdot 4\pi \int_0^\infty \frac{dk}{\left(\frac{2\pi}{\cancel{L}}\right)^3} k^2 \delta(\varepsilon - \varepsilon_k)$$

$$= \frac{1}{2\pi^2} \int_0^\infty dk \, k^2 \, \delta\left(\varepsilon - \frac{\hbar^2 k^2}{2m}\right)$$

$$= \frac{1}{2\pi^2} \int_0^\infty dk \, k^2 \cdot \frac{2m}{\hbar^2} \delta\left(k^2 - \frac{2m}{\hbar^2}\varepsilon\right)$$

$$= \frac{1}{2\pi^2} \int_0^\infty dk \, k^2 \frac{2m}{\hbar^2} \delta\left(\left(k + \frac{\sqrt{2m\varepsilon}}{\hbar}\right)\left(k - \frac{\sqrt{2m\varepsilon}}{\hbar}\right)\right)$$

$$= \frac{1}{2\pi^2} \cdot \frac{2m\boxed{\varepsilon}}{\hbar^2} \cdot \frac{2m}{\hbar^2} \cdot \frac{\cancel{\hbar}}{2\sqrt{2m\boxed{\varepsilon}}}$$

$$= \frac{1}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{\varepsilon}$$

と表せるから. $A = \frac{1}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}, \quad a = \frac{1}{2}$ //

(b) $\Omega = F - \mu N$

$$= \underline{\underline{k_BTN(3\log\lambda - \log\frac{k_BT}{P} - 1)}} - 3\underline{\underline{k_BTN(\log\lambda}} - \log\frac{k_BT}{P})$$

$$= 2k_BTN\log\frac{k_BT}{P} - k_BTN$$

$$= (2k_BT\log\frac{k_BT}{P} - k_BT)N$$

$\varepsilon - \mu > 0$

ここで N の表式

$\varepsilon > \mu$.

$$N = V\int_0^\infty \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{\sqrt{\varepsilon}}{e^{(\varepsilon-\mu)/k_BT} - 1} d\varepsilon$$

$$\frac{\partial N}{\partial V} = \int_0^\infty \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{\sqrt{\varepsilon}}{e^{(\varepsilon-\mu)/k_BT} - 1} d\varepsilon$$

Maxwell の関係式の両辺を μ で積分する：

$$(\text{左辺}) = \int_{-\infty}^{\mu} \frac{\partial P}{\partial \mu} d\mu = P(T, V, \mu) - P(T, V, \mu \to -\infty)$$

$$= P(T, V, \mu)$$

$$(\text{右辺}) = \int_{-\infty}^{\mu} \frac{\partial N}{\partial V} d\mu$$

$$= \int_{-\infty}^{\mu} d\mu \left\{ \int_0^\infty \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{\sqrt{\varepsilon}}{e^{(\varepsilon-\mu)/k_BT} - 1} d\varepsilon \right\}$$

$$= \int_0^\infty d\varepsilon \frac{\sqrt{\varepsilon}}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \underline{\underline{\int_{-\infty}^{\mu} \frac{d\mu}{e^{(\varepsilon-\mu)/k_BT} - 1}}}$$

$$\underline{\underline{\quad}} = \int_{-\infty}^{\mu} \frac{e^{-(\varepsilon-\mu)/k_BT}}{1 - e^{-(\varepsilon-\mu)/k_BT}} d\mu$$

$$= \left[-k_BT\log(1 - e^{-(\varepsilon-\mu)/k_BT}) \right]_{-\infty}^{\mu}$$

$$= -k_BT\log(1 - e^{-\beta(\varepsilon-\mu)})$$

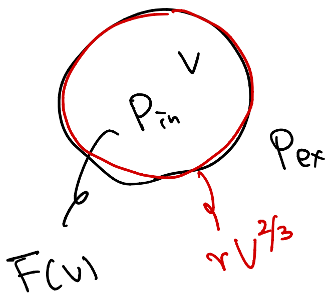
$$\therefore P(T, V, \mu) = -k_BT\int_0^\infty \frac{\sqrt{\varepsilon}}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \log(1 - e^{-\beta(\varepsilon-\mu)}) d\varepsilon$$

Ⅲ.

(7) ヘルムホルツの自由エネルギー $F(T,V,N)$ は、

$$dF = \underbrace{\left(\frac{\partial F}{\partial T}\right)_{V,N}}_{=-S} dT + \underbrace{\left(\frac{\partial F}{\partial V}\right)_{T,N}}_{=-P} dV + \underbrace{\left(\frac{\partial F}{\partial N}\right)_{T,V}}_{=\mu} dN$$

以上より $P = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$ がいえる。



$$P_{in} = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$$

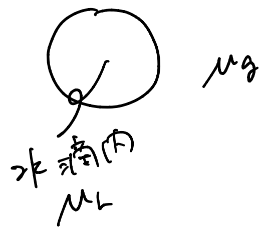
$$P_{ex} = -\left(\frac{\partial F_{tot}}{\partial V}\right)_{T,N} = -\left(\frac{\partial F}{\partial V}\right)_{T,N} - \frac{2}{3} r V^{-\frac{1}{3}}$$

以上より

$$\Delta P = P_{in} - P_{ex} = +\frac{2}{3} r V^{-\frac{1}{3}}$$

よって $B = +\frac{2}{3} r$, $b = -\frac{1}{3}$

IV



(8) $$P_L - P_G = \frac{2}{3} r V^{-\frac{1}{3}}$$

$$\frac{\partial P_L}{\partial V} - \frac{\partial P_G}{\partial V} = \frac{2}{3} \cdot \left(-\frac{1}{3}\right) r V^{-\frac{4}{3}}$$

$\mu_L = \mu_G$ より $$v_L dP_L = v_G dP_G$$

$$dP_L = \frac{v_G}{v_L} dP_G$$

$$\underline{\frac{v_G}{v_L} \frac{\partial P_G}{\partial V}} - \frac{\partial P_G}{\partial V} = -\frac{2}{9} r V^{-\frac{4}{3}}$$

$$\left(\frac{v_G}{v_L} - 1\right) \frac{\partial P_G}{\partial V} = -\frac{2}{9} r V^{-\frac{4}{3}}$$

$\frac{v_G}{v_L} \gg 1$ より $$\frac{v_G}{v_L} \frac{\partial P_G}{\partial V} = -\frac{2}{9} r V^{-\frac{4}{3}}$$ //

(9)

$$\frac{k_B T}{v_L} \frac{1}{P_G} dP_G = -\frac{2}{9} \gamma V^{-\frac{4}{3}} dV$$

$$\frac{k_B T}{v_L} \log P_G = \frac{2}{3} \gamma V^{-\frac{1}{3}} + \text{Const}$$

$$\log P_G = \frac{2\gamma v_L}{3 k_B T} V^{-\frac{1}{3}} + \text{Const}$$

$$P_G = A e^{\frac{2\gamma v_L}{3 k_B T} V^{-\frac{1}{3}}}$$

よって.

$$P_G = P_\infty e^{\frac{2\gamma v_L}{3 k_B T} V^{-\frac{1}{3}}}$$